MATHEMATICS

MATRICULATION

Standard - X

Untouchability is a sin
Untouchability is a crime
Untouchability is inhuman

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PREFACE

'Life is good for only two things-
Discovering Mathematics and Teaching Mathematics'

–Simeon Poisson

Now, for three things …… for Learning Mathematics too….

This revised edition of Mathematics textbook for X Standard Matriculation is prepared in conformity with the new guidelines and syllabi prescribed by the Directorate of Matriculation Schools, Government of Tamilnadu. This is an endeavour to make Mathematics – "Queen of Science" – as a favourite subject to all the learners. Since the subject has special significance with relevance to other branches of studies like Science and Technology, the book has been prepared in tune with worldwide standard.

Experts who are not merely masters in the subject, but have great insight on students' mentality prepared the manuscript. The draft materials were refined through mutual discussion and then exposed to a review workshop for incorporating valuable suggestions. The presentation is very lucid with appropriate illustrations and practical applications. It is hoped that the exercises given under each concept accelerate the process of acquiring the required aptitude to master the subject.

The book consists of chapters of both Paper I and II. Paper I includes Number Work, Mensuration, Set Language, Consumer Arithmetic, Algebra and Graph. In Paper II, Matrices, Theoretical Geometry, Co-ordinate Geometry, Trigonometry, Practical Geometry and Statistics are given.

Formulae are clearly arrived at and applied systematically to enhance easy understanding. Illustrations and diagrammatic explanations are aptly given to suit the concept discussed. There are about 463 examples with detailed steps to make the concepts more insightful and easier to assimilate.

We hope this book would prove to be an absolute elucidation to the users to solve problems. Learners should not restrict themselves to the given exercises only. They shall frame objective questions on their own and practice to get a wider knowledge and develop self-confidence to face Mathematical contests.

We wish the users to utilize this tool effectively and enjoy doing calculations intuitively. Suggestions and constructive criticisms from the users will be gratefully acknowledged.

Mathematics is not a spectators' sport
It is a "doing" subject.

Swaminathan .R
Chairperson

Contents

1. NUMBER WORK

1.0. INTRODUCTION

There is a saying highlighting the significance of numbers, as "***Numbers and Alphabets are the two eyes of human beings***". They form the base for any study. The former is infinite and the latter is finite. While the finite set of alphabets gives rise to ocean of words, just imagine about the infinite numbers. Though the invention of numbers comes first in the chronological evolution of Mathematics, it hasn't ever reached culmination, thus proving its marvellous power.

This branch of Mathematics dealing with study of numbers, also called as '***Arithmetics***' forms the heart of all other branches of Mathematics and is burgeoning everyday. When the world felt the need for a better number system, 'zero' appeared in India as early as 9^{th} century. In early English and American schools, the term 'ciphering' referred to doing sums or other computations in arithmetic. The field grows with time and the Mathematicians are incessantly adding feathers to its crown by numerous inventions and we are now provided with various Concepts and Theorems.

One such interesting concept invented by Leonardo of Pisa, later known as Fibonacci is '***Fibonacci Series***', which helps in the study of natural happenings like branching of plants, leaves and petal arrangements, etc. Likewise, various arrangements and patterns of numbers are invented for practical applications as theories.

Not less was the contribution of our Indian Mathematician Srinivasa Ramanujan. His crowning achievement was the discovery of Ramanujan Number '1729', by serendipity. The speciality of this magic number is that it is the smallest number, which could be expressed as sum of two cubes in two different ways.

(i.e.,) $9^3 + 10^3 = 1^3 + 12^3 = 1729$. Also,

(i) In the decimal representation of transcendental number 'e', 1729^{th} decimal place is the beginning of the first occurrence of all ten digits (0 to 9) consecutively.

(ii) The sum of the digits of 1729 is $1 + 7 + 2 + 9 = 19$; $19 \times 91 = 1729$

Subsequently, this concept gave rise to Euler's result :

$$59^4 + 158^4 = 133^4 + 134^4 = 635318657$$

God invented the integers ; all else is the work of man – ***Kronecker***

I know numbers are beautiful. If they aren't beautiful, nothing is – ***Erdos***

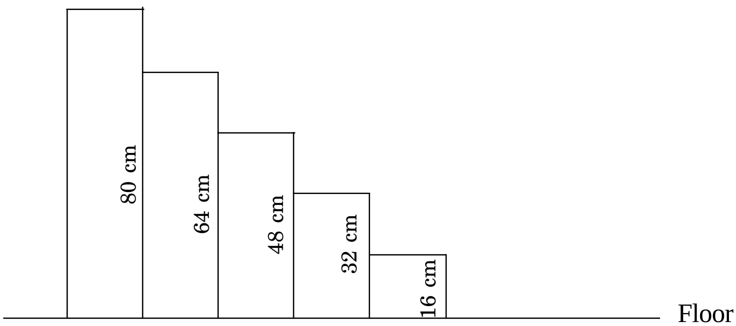
Thus, Mathematics is the **Queen of Science** and Number Theory is the **Queen of Mathematics.** The wonders of numbers are numerous and amazing. In this chapter, we shall discuss some concepts of Number Work.

1.1. SEQUENCE AND SERIES

1.1.1. Introduction

We often come across numbers that follow a particular pattern. In Mathematics, patterns play an important role. From them we may make generalisation and establish procedures to apply in different fields of study.

Let us consider a very simple example. As we walk up or down stairs, we sometimes count the steps unconsciously. Have we ever thought of how much high each step is? Suppose the rise is 16 cm and if we are to make a record of the heights above the floor of the steps taken in order, we would get 16 cm, 32 cm, 48 cm, 64 cm, 80 cm and so on. Refer the figure given below;



We get the following numbers: 16, 32, 48, 64, 80

In the above arrangement, numbers are arranged in a definite order according to some rule. (i.e.,) It starts with 16 and ends with 80 and each number is obtained from the previous one by adding 16 to it.

Let us consider the following arrangements of numbers.

1, 4, 9, 16, 25, ...

$1, \frac{1}{8}, \frac{1}{27}, \frac{1}{64}, \ldots$

1, 3, 5, 7, 9, ...

In the above arrangements of numbers, there are some rules.

In the first arrangement the numbers are squares of natural numbers and in the second arrangement the numbers are reciprocals of cubes of natural numbers whereas in the third arrangement the numbers are odd natural numbers.

Here, we find the numbers arranged according to some specific rule and this helps us to find out other numbers that follow. Such an arrangement is called a **Sequence**. In this chapter, we shall study about sequences and particular types of sequences called Arithmetic Sequence and Geometric Sequence.

Thus, we may define a sequence formally as follows.

1.1.2. Sequence

Definition: A sequence is an arrangement of numbers in a definite order according to some rule.

(i.e.,) An arrangement showing a relation between the numbers as determined by some rule is called a sequence.

There are numerous ways to relate numbers to each other according to some definite rule;

(a) By adding a number
2, 7, 12, 17, 22, ... (using 5)

(b) By subtracting a number
7, 4, 1, –2, –5, ... (using 3)

(c) By multiplying by a number
2, 8, 32, 128, 512, ... (using 4)

(d) By dividing by a number
$27, -9, 3, -1, \frac{1}{3}, \ldots$ (using –3)

(e) By multiplying by a number and adding a number
1, 6, 16, 36, 76, ... (multiplying by 2 and adding 4)

1.1.3. Terms of Sequence

The various numbers occurring in a sequence are called its terms. We denote the terms of a sequence by a_1, a_2, a_3, ... Here, the subscripts denote the position of the terms.

The number at the first place is called the first term of the sequence and is denoted by a_1. The number at the second place is called the second term of the sequence and is denoted by a_2 and so on.

In general, the number at the n^{th} place is called the n^{th} term of the sequence and is denoted by a_n. The n^{th} term is also called the general term of the sequence.

For example, $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$ is a sequence whose first term is $\frac{1}{2}$

(i.e.,) $a_1 = \frac{1}{2}$ and 2^{nd} term $a_2 = \frac{2}{3}$

Rule of a Sequence in Terms of Algebraic Formula :

The rule which generates the various terms of a sequence can be expressed as an algebraic formula.

For example, consider the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

Here, $a_1 = \frac{1}{2} = \frac{1}{1+1}$; $a_2 = \frac{2}{3} = \frac{2}{2+1}$; $a_3 = \frac{3}{4} = \frac{3}{3+1}$; $a_4 = \frac{4}{5} = \frac{4}{4+1}$

Proceeding like this, in general, $a_n = \frac{n}{n+1}$

Thus, the above sequence is described as $a_n = \frac{n}{n+1}$ for $n \geq 1$ which is the algebraic formula for the rule which generates the above sequence.

1.1.4. Real Sequence

If for every positive integer 'n' there is a unique real number a_n according to some rule, then the ordered set of real numbers $a_1, a_2, a_3, \ldots, a_n$ is said to define a real sequence.

1.1.5. Finite Sequence and Infinite Sequence:

Finite Sequence: A sequence which has a finite number of terms is called a finite sequence. *Example* : 3, 4.5, 6, 7.5, 9, 10.5, 12, 13.5.

This sequence has a finite number of terms. (i.e.,) it has a last term.

$\therefore$ This sequence is called a finite sequence.

Infinite Sequence: A sequence which has infinite number of terms is called an infinite sequence. *Example* : 8, 4, 0, – 4, – 8, – 12, ...

This sequence has infinite number of terms. (i.e.,) it has no last term.

$\therefore$ This sequence is called an infinite sequence.

1.1.6. Series:

For each sequence there is an associated series, which is obtained by replacing the commas in the ordered set of terms by plus symbols.

(i.e.,) When the terms of a sequence are connected by plus signs, they are said to form a **Series**.

(i.e.,) If $a_1, a_2, a_3, \ldots$ is a sequence, then $a_1+a_2+a_3+\ldots$ is the series corresponding to the given sequence.

Example 1.1 : Give the first 3 terms of the sequence $a_n = 2^n - 1,\ n \in N$

Solution : Putting n = 1, 2, 3 in $a_n = 2^n - 1$

We get, $a_1 = 2^1 - 1 = 1$

$a_2 = 2^2 - 1 = 3$

$a_3 = 2^3 - 1 = 7$

$\therefore$ The first three terms of the sequence are 1, 3, 7.

Example 1.2 : $1, \frac{1}{2}, \frac{1}{3},$ ... is a sequence. Find the n^{th} term of the sequence.

Solution : Here, $a_1 = 1 = \frac{1}{1}$; $a_2 = \frac{1}{2}$; $a_3 = \frac{1}{3}$

By observation, the n^{th} term of the sequence is given by $a_n = \frac{1}{n}$.

Example 1.3 : Write the series corresponding to the sequence whose n^{th} term is $a_n = (n+1)(n+2)$.

Solution : By the given data $a_n = (n + 1)(n + 2)$

Putting n = 1, 2, 3, ... we get,

$a_1 = (1+1)(1+2) = 6$

$a_2 = (2+1)(2+2) = 12$

$a_3 = (3+1)(3+2) = 20$

In the same manner, remaining terms of the sequence can be generated. Thus the given sequence is 6, 12, 20, ... and the corresponding series is 6 + 12 + 20+ ...

EXERCISE 1.1

1. Write the first three terms of the sequences whose general terms are given below;

   (i) $4n+1$ (ii) $\frac{1-n-n^2}{(-1)^n}$ (iii) n^2-1 (iv) $\frac{(-1)^n}{n}$

2. Find the n^{th} term of the following sequences;

   (i) 2, 5, 10, ... (ii) $\frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \cdots$ (iii) $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \cdots$ (iv) 1, 3, 7, 15,...

3. If $a_1 = -1$, $a_n = \frac{a_{n-1}}{n}$; $n > 1$, find a_4 and a_5.

4. Find the 3rd and 4th terms of the sequence, if $a_n = \begin{cases} n^2 & if \quad n \quad is \quad odd \\ \frac{n^2}{2} & if \quad n \quad is \quad even \end{cases}$

5. Write the series corresponding to the sequences;

   (i) 3, 8, 15, 19, ... (ii) General term of the sequence is $a_n = (-1)^{n+1}\, 3^{n-1}$

   (iii) General term of the sequence is $a_n = 4n^2 + 3$

   (iv) General term of the sequence is $a_n = 2n^2 - 1$

1.2. ARITHMETIC PROGRESSION (A.P)

1.2.1. Arithmetic Sequences:

Consider the following sequences;

(i) 2, 6, 10, 14, 18, ...

(ii) $-9, -5, -1, +3, +7, \ldots$

(iii) $\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, \ldots$

In (i) $6 - 2 = 10 - 6 = 14 - 10 = 18 - 14 = 4$

In (ii) $(-5) - (-9) = (-1) - (-5) = 3 - (-1) = 7 - 3 = 4.$

In (iii) $\frac{3}{2} - \frac{5}{2} = \frac{1}{2} - \frac{3}{2} = \left(-\frac{1}{2}\right) - \frac{1}{2} = -1$

We observe that in these sequences, each term after the first term differs from the preceding term by a constant amount. These types of sequences are called Arithmetic sequences.

Progression: Sequences following certain patterns are known as progressions. In this case, the pattern follows certain Mathematical rule. All progressions are sequences, but the converse is not true.

Example : Consider the sequence of prime numbers 2,3,5,7,11,13,... . For this sequence nobody can find out the definite pattern following which each number can be obtained from the previous number of the sequence.

$\therefore$ 2, 3, 5, 7, 11, 13, ... is a sequence but not a progression.

Now, consider the sequence 2, 4, 6, 8, ... In this sequence, each number can be obtained from the previous number by adding it with 2.

$\therefore$ This sequence 2, 4, 6, 8, ... is called a **Progression**.

Example: The following sequences are progressions:

(i) 1, 5, 9, 13, ...

(ii) 10, 8, 6, 4, ...

(iii) $1, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}, \ldots$

The above Arithmetic sequences have certain patterns.

$\therefore$ Arithmetic sequence is also known as Arithmetic Progression.

Arithmetic Progression (A.P):

A sequence is said to be an Arithmetic Progression (abbreviated as A.P) if the difference of each term, except the first one, from its preceding term is always same.

(i.e.,) A sequence $a_1, a_2, a_3, \ldots$ is said to be an Arithmetic Progression if $a_{n+1} - a_n$ = constant for all $n \in \mathbb{N}$.

The constant difference, generally denoted by '*d*' is called the common difference.

Examples : 5, 8, 11, 14, ... is an A.P whose first term is 5 and the common difference is 3.

16, 12, 8, 4, 0, – 4, ... is an A.P whose first term is 16 and the common difference is – 4.

General Form of an A.P:

The first term of an A.P is generally denoted by '*a*' and the common difference is denoted by '*d*'.

1^{st} term $= a$
2^{nd} term $= a + d$
3^{rd} term $= (a + d) + d = a + 2d$

$\therefore$ General form of an A.P with first term 'a' and common difference 'd' is $a, a + d, a + 2d, a + 3d, ...$

Remark: If we add the common difference to any term of A.P, we get the next following term and if we subtract it from any term, we get the preceding term.

General Term of an A.P:

In the A.P $a, a + d, a + 2d, a + 3d, ...$

1^{st} term $= t_1 = a = a + (1 - 1)d$
2^{nd} term $= t_2 = a + d = a + (2 - 1)d$
3^{rd} term $= t_3 = a + 2d = a + (3 - 1)d$
4^{th} term $= t_4 = a + 3d = a + (4 - 1)d$

Proceeding like this, we get the n^{th} term as $t_n = a + (n - 1)d$

$\therefore$ The general term or n^{th} term of an A.P. is $t_n = a + (n - 1)d$.

Last Term & Number of Terms of a Finite A.P:

The n^{th} term t_n is also known as the last term. It is denoted by l.

Now $l = a + (n - 1)d \Rightarrow n = \frac{l - a}{d} + 1$

This formula helps us to find the number of terms of a finite A.P.

Properties of an A.P:

(i) If a constant quantity is added to or subtracted from each term of a given A.P, we get another A.P.

17, 20, 23, ... is an A.P with $d = 3$. Add 5 to each term of the above sequence. We get the sequence 22, 25, 28, ... which is also an A.P with $d = 3$.

Subtract 4 from each term of the given A.P. The resulting sequence 13, 16, 19, ... is also an A.P with $d = 3$.

(ii) If each term of a given A.P is multiplied or divided by a non-zero constant, another A.P is formed.

5, 10, 15, 20, ... is an A.P with $d = 5$.

Multiply each term by 2. We get, 10, 20, 30, 40, ... which is also an A.P with $d = 10$.

Divide each term by 5. We get, 1, 2 ,3, 4,...which is also an A.P with $d = 1$.

Example 1.4 : Show that the sequence described by $a_n = \frac{1}{3}n + \frac{1}{6}$ is an A.P.

Solution : By the given data, $a_n = \frac{1}{3}n + \frac{1}{6}$

Putting $n = 1, 2, 3, \ldots$ we get, $a_1 = \frac{1}{3} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2} = t_1$ (say)

$a_2 = \frac{2}{3} + \frac{1}{6} = \frac{5}{6} = t_2$ (say); $a_3 = \frac{3}{3} + \frac{1}{6} = \frac{7}{6} = t_3$ (say)

Here $t_2 - t_1 = \frac{5}{6} - \frac{1}{2} = \frac{2}{6} = \frac{1}{3}$; $t_3 - t_2 = \frac{7}{6} - \frac{5}{6} = \frac{2}{6} = \frac{1}{3}$

$\therefore \ t_2 - t_1 = t_3 - t_2 = \frac{1}{3}$

$\therefore$ The given sequence is an A.P with common difference $\frac{1}{3}$.

Example 1.5 : The first term of an A.P is 6 and the common difference is 5. Find the A.P and its general term.

Solution : By the given data, $a = 6$, $d = 5$.

The general form of an A.P is $a, a+d, a+2d, \ldots$

The required A.P is 6, 6 + 5, 6 + 2(5), ...

(i.e.,) 6, 11, 16, ...

The general term $t_n = a + (n-1)d$

$= 6 + (n-1)5 \qquad = 6 + 5n - 5$

$\therefore t_n = 1 + 5n$

Example 1.6 : Find the common difference and 10th term of the A.P 100, 96, 92, ...

Solution : From the given A.P $a = 100$; $d = t_2 - t_1 = 96 - 100 = -4$

We know that $t_n = a + (n-1)d$

$t_{10} = a + 9d = 100 + 9(-4) = 100 - 36 = 64$

Example1.7 : Which term of the sequence 13, 15, 17, ... is 71?

Solution : Here $a = 13$, $d = 15 - 13 = 2$ and $l = 71$.

We know that $n = \left(\frac{l-a}{d}\right) + 1$

$$n = \left(\frac{71-13}{2}\right) + 1 = \frac{58}{2} + 1 = 30$$

$\therefore$ 71 is the 30^{th} term.

Example 1.8: Find the middle term of an A.P 3, 5, 7, ... , 71.

Solution : Here $a = 3$, $d = 2$ and $l = 71$

We know that $n = \left(\frac{l-a}{d}\right) + 1$

$$n = \left(\frac{71-3}{2}\right) + 1 = \frac{68}{2} + 1 = 35$$

$\therefore$ Total number of terms in the given A.P is 35.

$\therefore$ Middle term in the A.P is $\left(\frac{35+1}{2}\right)^{th}$ term

(i.e.,) Middle term $= 18^{th}$ term

$= t_{18} = a + 17d = 3 + 17\,(2) = 3 + 34 = 37$

$\therefore$ The middle term = 37

Example 1.9: The 5^{th} term of an A.P is 27 and the 8^{th} term is 12. Determine the A.P.

Solution : Given that $t_5 = 27$ and $t_8 = 12$

(i.e.,) $t_5 = a + 4d = 27$ ....(1)

and $t_8 = a + 7d = 12$ ....(2)

(2) – (1) $\Rightarrow$ $d = -5$

Substituting $d = -5$ in (1), we get,

$a + 4\,(-5) = 27$

$a = 27 + 20 = 47$

The required A.P is 47, 47 + (–5), 47 + 2 (–5), ...

(i.e.,) 47, 42, 37, ...

Example 1.10: For what value of n, the n^{th} term of the series 3 + 10 + 17 +... and 63 + 65 + 67 + ... are equal?

Solution : In the I series, we have $a = 3$, $d = 10 - 3 = 7$

$$t_n = a + (n - 1)\, d = 3 + (n - 1)\, 7 = 7n - 4$$

In the II series, we have $a = 63$, $d = 65 - 63 = 2$

$$t_n = a + (n - 1)\, d = 63 + (n - 1)\, 2 = 2n + 61$$

By the given data, n^{th} terms of the given series are equal

$$\Rightarrow\ 7n - 4 = 2n + 61 \Rightarrow 5n = 65 \ \Rightarrow\ n = 13$$

Example 1.11: Which term of the A.P 19, $18\frac{1}{5}$, $17\frac{2}{5}$, ... is the first negative term?

Solution : The given Arithmetic Progression is $\frac{95}{5}, \frac{91}{5}, \frac{87}{5}, \ldots$

By the property of A.P, consider the following A.P 95, 91, 87, ...

Here $a = 95$ and $d = -4$

Let the n^{th} term of the given A.P be the first negative term. Then,

$$t_n < 0$$

$$\Rightarrow\ a + (n - 1)\, d < 0$$

$$\Rightarrow\ 95 + (n - 1)(-4) < 0$$

$$\Rightarrow\ 99 - 4n < 0$$

$$\Rightarrow\ 99 < 4n$$

$$\Rightarrow\ n > \frac{99}{4}$$

$$\Rightarrow\ n > 24.75$$

Since 25 is the natural number just greater than 24.75, $n = 25$

Thus, 25^{th} term of the given A.P is the first negative term.

Example 1.12 : The sum of three numbers is 54 and their product is 5670. Find the number.

Solution : Let the three numbers in A.P be $a - d, a, a + d$

Given the sum of three numbers $= 54$

(i.e.,) $(a - d) + a + (a + d) = 54$

$$3a = 54$$

$$a = 18$$

Also, given their product $= 5670$

(i.e.,) $(a - d)(a)(a+d) = 5670$

$\Rightarrow \quad a(a^2 - d^2) = 5670$

$\Rightarrow \quad 18(18^2 - d^2) = 5670$

$\Rightarrow \quad 18(324 - d^2) = 5670$

$\Rightarrow \quad d = \pm 3$

When $a = 18$, $d = 3$ the required 3 numbers are 15, 18, 21

When $a = 18$, $d = -3$ the required 3 numbers are 21, 18, 15

Example1.13: Find four numbers in A.P whose sum is 20 and the sum of whose squares is 120.

Solution : Let the four numbers be $a - 3d, a - d, a + d, a+3d$

Their sum $= 20$

$\therefore (a - 3d) + (a - d) + (a + d) + (a + 3d) = 20$

$\Rightarrow \quad 4a = 20$

$\Rightarrow \quad$ (i.e.,) $a = 5$

Also given, the sum of squares $= 120$

$\Rightarrow (a - 3d)^2 + (a - d)^2 + (a + d)^2 + (a + 3d)^2 = 120$

$\Rightarrow (5 - 3d)^2 + (5 - d)^2 + (5 + d)^2 + (5 + 3d)^2 = 120 \quad [\because a = 5]$

$\Rightarrow \quad 20d^2 = 20$

$\Rightarrow \quad d = \pm 1$

When $a = 5$, $d = 1$

The required four numbers are $5 - 3(1), 5-1, 5+1, 5+3(1)$

(i.e.,) 2, 4, 6, 8

When $a = 5$, $d = -1$

The required four numbers are $5-3(-1), 5-(-1), 5+(-1), 5+3(-1)$

(i.e.,) 8, 6, 4, 2

Example1.14: If $(p + 1)^{th}$ term of an A.P is twice $(q + 1)^{th}$ term, prove that the $(3p+1)^{th}$ term is twice the $(p+q+1)^{th}$ term.

Solution : Let the first term of the A.P be 'a' and the common difference be 'd'

We have $\quad t_{p+1} = 2t_{q+1}$

$$\Rightarrow \quad a + (p+1-1)\,d = 2\,[\,a + (q+1-1)\,d\,]$$
$$\Rightarrow \quad a + pd = 2[\,a + qd\,]$$
$$\Rightarrow \quad a + pd = 2a + 2qd$$
$$\Rightarrow \quad a = (p - 2q)\,d \qquad \text{..............} \ (1)$$

Now, $t_{3p+1} = a + (3p + 1 - 1)\,d$

$= (p - 2q)d + 3\,pd = 4pd - 2qd$

$\therefore \ t_{3p+1} = 2[\,2\,pd - qd\,]$ .... (2)

and $t_{p+q+1} = a + (p + q + 1 - 1)\,d$

$= (p - 2q)\,d + (p + q)\,d$ from (1)

$t_{p+q+1} = (2p - q)\,d$

$t_{p+q+1} = 2pd - qd$ .... (3)

From (2) and (3) $t_{3p+1} = 2t_{p+q+1}$

Example1.15: If a^2, b^2, c^2 are in A.P, show that $\dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}$ are also in A.P.

Proof : Given a^2, b^2, c^2 are in A.P.

$$b^2 - a^2 = c^2 - b^2$$
$$\Rightarrow \quad (b + a)(b - a) = (c + b)(c - b)$$
$$\Rightarrow \quad (b + a)(b + c - c - a) = (c + b)(c + a - a - b)$$
$$\Rightarrow \quad (b + a)(b + c - c - a) = (c + b)(c + a - a - b)$$
$$\Rightarrow \quad (b + a)[(b + c) - (c + a)] = (c + b)[(c + a) - (a + b)]$$
$$\Rightarrow (a + b)(b + c) - (a + b)(c + a) = (b + c)(c + a) - (b + c)(a + b)$$

Dividing both sides by $(b + c)(c + a)(a + b)$, we get,

$$\frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a} \quad [\because b+c \neq 0 \text{ and } c+a \neq 0 \text{ and } a+b \neq 0]$$

$$\Rightarrow \quad \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \text{ are also in A.P.}$$

Example 1.16: If $\dfrac{1}{p}, \dfrac{1}{q}, \dfrac{1}{r}$ are in A.P and $p + q + r \neq 0$, show that $\dfrac{q+r}{p}, \dfrac{r+p}{q}, \dfrac{p+q}{r}$ are also in A.P.

Proof : Given $\dfrac{1}{p}, \dfrac{1}{q}, \dfrac{1}{r}$ are in A.P. Multiply each term by $p + q + r \ (\neq 0)$

$\Rightarrow \quad \dfrac{p+q+r}{p}, \dfrac{p+q+r}{q}, \dfrac{p+q+r}{r}$ are also in A.P. (by the property of A.P.)

$\Rightarrow \quad 1+\dfrac{q+r}{p}, 1+\dfrac{p+r}{q}, \dfrac{p+q}{r}+1$

Subtracting '1' from every term of the above, we get, $\dfrac{q+r}{p}, \dfrac{p+r}{q}, \dfrac{p+q}{r}$

which is also an A.P. (by the property of A.P)

Example 1.17 : If $a^x = b^y = c^z$ and $b^2 = ac$, show that $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$ are in A.P.

Proof : Given that $a^x = b^y = c^z = m$ (say)

(i.e.,) $\quad a = m^{\frac{1}{x}} \quad b = m^{\frac{1}{y}} \quad c = m^{\frac{1}{z}}$

but $b^2 = ac$ (given)

Then , $\left(m^{1/y}\right)^2 = \left(m^{1/x}\right)\left(m^{1/z}\right)$

$$\Rightarrow \quad \left(m^{2/y}\right) = \left(m^{1/x + 1/z}\right)$$

$$\Rightarrow \quad \frac{2}{y} = \frac{1}{x} + \frac{1}{z}$$

$\Rightarrow \quad \dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$ are in A.P.

Example 1.18: If $a_1, a_2, a_3, \ldots, a_n$ be an A.P of non-zero terms, prove that

$$\frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \ldots + \frac{1}{a_{n-1}a_n} = \frac{n-1}{a_1a_n}$$

Solution : Let *'d'* be the common difference of the given A.P.

$$\therefore d = a_2 - a_1 = a_3 - a_2 = \cdots = a_n - a_{n-1}$$

$$\frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \ldots + \frac{1}{a_{n-1}a_n} = \frac{1}{d}\left[\frac{d}{a_1a_2} + \frac{d}{a_2a_3} + \ldots + \frac{d}{a_{n-1}a_n}\right]$$

$$= \frac{1}{d}\left[\frac{a_2 - a_1}{a_1 a_2} + \frac{a_3 - a_2}{a_2 a_3} + \ldots + \frac{a_n - a_{n-1}}{a_{n-1} a_n}\right]$$

$$= \frac{1}{d}\left[\left(\frac{1}{a_1} - \frac{1}{a_2}\right) + \left(\frac{1}{a_2} - \frac{1}{a_3}\right) + \ldots + \left(\frac{1}{a_{n-1}} - \frac{1}{a_n}\right)\right]$$

$$= \frac{1}{d}\left[\frac{1}{a_1} - \frac{1}{a_n}\right] = \frac{a_n - a_1}{d a_1 a_n} = \frac{a_1 + (n-1)d - a_1}{d a_1 a_n} = \frac{n-1}{a_1 a_n}$$

EXERCISE 1.2

1. Which of the following sequences are in A.P?

   (i) $\frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \cdots$ (ii) $\frac{1}{4}, \frac{7}{12}, \frac{11}{12}, \cdots$ (iii) $1^2, 2^2, 3^2, 4^2, \cdots$

   (iv) $a, a, a, a, \cdots$ (v) $m, m+3, m+6, m+9, \cdots$

2. (i) Which term of the sequence 17, 20, 23, ... is 56?

   (ii) Is 68 a term of the sequence 7, 10, 13, ... ?

   (iii) For what value of n, the n^{th} terms of the A.Ps 9,7,5,...and 15,12,9, ...are same?

3. (i) For the A.P 45, 41, 37, ... , find t_{10} and t_{n+1} .

   (ii) Which term of the sequence 24, $23\frac{1}{4}$, $22\frac{1}{2}$, $21\frac{3}{4}$, ... is the first negative term?

   (iii) Find the 8^{th} term of an A.P whose 15^{th} term is 49 and 19^{th} term is 65.

   (iv) Find the 40^{th} term of an A.P where $t_8 = 56$ and $t_{15} = 91$.

   (v) Find the 6^{th} term from the end of the A.P 17, 14, 11, ...,–10.

4. (i) Find the middle term of an A.P with 23 terms, given its first term is 11 and common difference is 3.

   (ii) Find the middle term of an A.P –3, –1, 1, ... , 33.

5. (i) The fourth term of an A.P is equal to 3 times the first term, and the seventh term exceeds twice the 3^{rd} term by 1. Find the first term and the common difference.

   (ii) Find the A.P whose 3^{rd} term is 16 and the 7^{th} term exceeds its 5^{th} term by 12.

   (iii) Which term of an A.P 3, 15, 27, 39, ... is 132 more than its 54^{th} term?

6. (i) Divide 69 into three parts which are in A.P such that the product of the first two is 483.

(ii) Divide $12\frac{1}{2}$ into five parts which are in A.P such that the first and the last parts are in the ratio 2 : 3

7. (i) The 8^{th} term of an A.P is zero. Prove that its 38^{th} term is triple its 18^{th} term.

   (ii) If 10 times the 10^{th} term of an A.P is equal to 20 times its 20^{th} term, show that the 30^{th} term of the A.P is zero.

8. (i) If the angles of a triangle are in A.P and the tangent of the smallest angle is 1, then find the other angles of the triangle.

   (ii) Let the angles A, B, C of a triangle ABC be in A.P and let B:C = 4 : 3. Find the angle A.

9. (i) Find the 3 numbers in A.P whose sum is 24 and the sum of their cubes is 1968.

   (ii) The sum of the 3 numbers is 12 and the sum of their squares is 56. Find the numbers.

   (iii) Find the 4 numbers in A.P whose sum is 20 and the sum of whose squares is 120.

10. The digits of a positive integer having three digits are in A.P and their sum is 15. The number obtained by reversing the digits is 594 less than the original number. Find the number.

11. Prove that in an A.P the sum of the terms equidistant from the beginning and the end is always same and equal to the sum of the first and last terms.

12. If the p^{th} term of an A.P is $\frac{1}{q}$ and q^{th} term is $\frac{1}{p}$, show that its $(pq)^{th}$ term is 1.

13. Given that $(p+1)^{th}$ term of an A.P is twice the $(q+1)^{th}$ term, prove that the $(3p+1)^{th}$ term is twice the $(p+q+1)^{th}$ term.

14. If $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P, show that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in A.P.

1.3. ARITHMETIC SERIES

1.3.1. Sum to n- Terms of an Arithmetic Progression:

The method used to find the sum of an A.P is known as Gaussian method of addition, named after the great German Mathematician, Karl Friedrich Gauss (1777 - 1855A.D). When Gauss was studying in elementary school, his teacher asked the students to find the sum 1 + 2 + 3 + ... + 100. While his classmates were trying to find the sum by actual addition, Gauss gave the answer immediately. His teacher was astonished by his intelligence. He wrote the numbers in the reverse order and added with the numbers in the natural order. (i.e.,)

$$\begin{array}{cccccccccccc} 1 & + & 2 & + & 3 & + & ... & + & 99 & + & 100 \\ 100 & + & 99 & + & 98 & + & ... & + & 2 & + & 1 \\ \hline 101 & + & 101 & + & 101 & + & ... & + & 101 & + & 101 \\ \hline \end{array}$$

He got 101 + 101+ ... +101 (i.e.,) 101 is got 100 times. So the sum is 100 × 101. He had used each number twice and so he divided the sum by 2. Thus he got, $\frac{100\times101}{2} = 5050$. Are you not astonished !

The same procedure is followed to find the sum of the first *n*- terms of an A.P $a, a+d, a+2d, ...$

Let us denote the sum upto *n*-terms by S_n

Then $S_n = a + (a+d) + (a+2d) + ...+ [a + (n-2) d] + [a + (n -1) d]$....... (1)

Write the above series in the reverse order. We get,

$S_n = [a + (n -1) d] + [a + (n -2) d] + ...+ (a+2d)+ (a+d) + a$........ (2)

Adding (1) and (2) we get,

$2S_n = [a + \{a + (n-1) d\}] + [(a+d) +\{a + (n-2)d \}] +...$

$+ [\{a + (n-2)d \}+ (a+2d)] + [\{a+(n-1)d\}+a]$

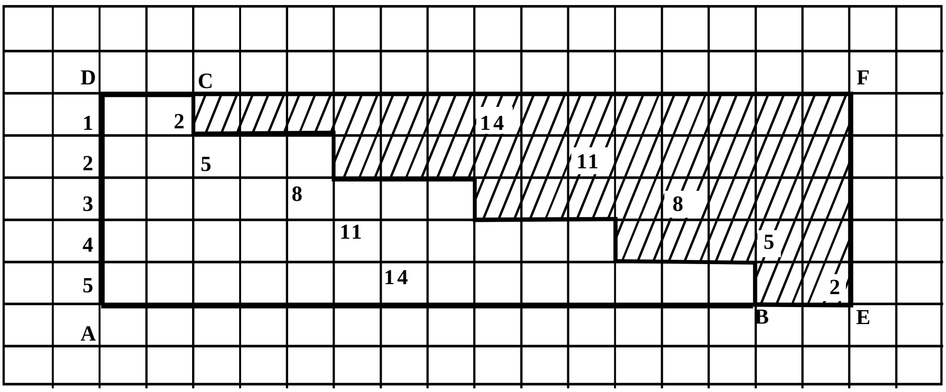
$2S_n = [2a +(n -1) d] + [2a +(n-1) d]+ ...+[2a +(n-1)d] + [2a+(n-1)d]$

$2S_n = [2a + (n-1) d] + [2a + (n-1) d] + ...$ to n terms.

$2S_n = n [2a + (n-1) d]$

$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$ or $S_n = \frac{n}{2} [a + l]$

The Visual Aid: The diagram ABCD in the figure depicts the Arithmetic Progression 2, 5, 8, 11, 14.



In order to determine the sum of its terms, complete the rectangle AEFD. Now, we get two equal figures ABCD and BEFC.

The area of each figure decribes the sum of our A.P. Hence the double sum of the progression is equal to the area of the rectangle AEFD.

Let us denote the sum of the given A.P by S.

Double sum = $2S$ = Area of the rectangle AEFD

= AE ×AD = (AB+BE) × AD

= (fifth term + first term) (number of terms in A.P)

$2S = (14 + 2)(5)$

$$S = \frac{5}{2}(2+14) \text{ which can be compared to } S_n = \frac{n}{2}(a+l)$$

Example1.19: Find the sum of the series 1+ 3 + 5 + ... + 399.

Solution: Here $a = 1$, $d = 2$, $l = 399$

$$\Rightarrow \qquad n = \left(\frac{399-1}{2}\right) + 1 = 200$$

$$S_{200} = \frac{200}{2}[1+399] = 100 \times 400 = 40000$$

Example1.20 : Sum of the series $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots$ to $2n$ terms.

Solution : $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots$ to $2n$ terms.

= $(1-4) + (9-16) + (25-36) + \ldots$ to n terms is an A.P with $a = -3$, $d = -4$

$$\therefore \text{ The required sum } = \frac{n}{2}[2(-3) + (n-1)(-4)] = \frac{n}{2}[-6 - 4(n-1)]$$

$$= \frac{n}{2}[-2-4n] = -n(2n+1)$$

Example1.21: Find the sum of all three digit numbers which are divisible by 9.

Solution: Given that the three digit numbers which are divisible by 9 are

108, 117, 126, ..., 999. Here $a = 108$, $d = 9$, $l = 999$

$$\Rightarrow \qquad n = \left[\frac{l-a}{d}\right] + 1 = \left[\frac{999-108}{9}\right] + 1 = 100$$

$$\therefore \quad S_{100} = \frac{100}{2}[108+999] = 55350$$

Example 1.22: In an A.P the sum of the first 11 terms is 44 and that of the next 11 terms is 55. Find the A.P.

Solution: Given $S_{11} = 44$

$\Rightarrow \quad \frac{11}{2}[\,2a + 10d\,] = 44$

$\Rightarrow \quad 2a + 10d = 8 \quad \text{.................(1)}$

Also given, the sum of the next 11 terms is 55

$\therefore$ The sum of first 22 terms = 55 + 44

$S_{22} = 99$

$\Rightarrow \quad = \frac{22}{2}[\,2a + 21d\,] = 99$

$\Rightarrow \quad 2a+21d = 9 \quad \text{..............(2)}$

(2) − (1) $\Rightarrow \quad 11d = 1$

$\Rightarrow \quad d = \frac{1}{11}$

Substituting $d = \frac{1}{11}$ in (1), we get,

$2a + 10\,d = 8$

$\Rightarrow \; 2a + \left(10 \times \frac{1}{11}\right) = 8 \quad \Rightarrow \; 2a = 8 - \frac{10}{11} \quad \Rightarrow \; \text{a} = \frac{39}{11}$

$\therefore$ The required A.P is $\frac{39}{11}, \frac{39}{11} + \frac{1}{11}, \frac{39}{11} + 2\left(\frac{1}{11}\right), \ldots$

(i.e.,) $\frac{39}{11}, \frac{40}{11}, \frac{41}{11}, \ldots$

Example 1.23: How many terms of the series 54, 51, 48, ... be taken so that their sum is 513? Explain the double answer.

Solution : The given sequence is an A.P with first term a = 54 and common difference d = −3.

Let the sum of n terms be 513. Then $S_n = 513$

$\Rightarrow \quad \frac{n}{2}[\,2a + (n-1)\,d\,] = 513$

$\Rightarrow \quad \frac{n}{2}[\,108 + (n-1)(-3)\,] = 513$

$\Rightarrow \quad n^2 - 37n + 342 = 0$

$\Rightarrow \quad (n - 18)(n - 19) = 0$

$\Rightarrow \quad n = 18$ (or) 19

Here the common difference is negative. So, 19th term is

$$t_{19} = 54 + (19 - 1)(-3)$$

$$= 54 - 54 = 0$$

Thus, the sum of first 18 terms as well as that of first 19 terms is 513.

Example1.24: Find the sum of first 20 terms of A.P in which 3rd term is 7 and 7th term is two more than thrice of its 3rd term.

Solution : Given $t_3 = 7$

$\Rightarrow \quad a + 2d = 7$ .... (1)

Also given, $t_7 = 3t_3 + 2$

$\Rightarrow \quad a + 6d = 3(7) + 2 = 23$ .... (2)

Subtracting (1) from (2), we get, $4d = 16 \Rightarrow d = 4$

Putting $d = 4$ in (1), we get, $a + 8 = 7 \Rightarrow a = -1$

$$\Rightarrow S_{20} = \frac{20}{2}[2 \times (-1) + (20 - 1)4] = 10(-2 + 76) = 740$$

Example 1.25 : A man arranges to pay off a debt of Rs.900 in 18 monthly instalments of Rs.50 each and interest at the rate of 5% per annum in the 19th instalment. Calculate the simple interest that he pays in the 19th instalment.

Solution: Given that, during the 1st instalment he has to pay interest on Rs. 900 which is equal to,

$$I_1 = \frac{900 \times 1 \times 5}{12 \times 100} \quad \text{................ (1)}$$

Principal amount for the second instalment will be Rs.850. (i.e.,) 900 –50

$$\therefore \text{ Interest } = I_2 = \frac{850 \times 1 \times 5}{12 \times 100} \quad \text{............... (2)}$$

Similarly, the last instalment will be just Rs.50

$$\therefore \text{ Interest } = I_{18} = \frac{50 \times 1 \times 5}{12 \times 100} \quad \text{............... (3)}$$

Thus, the amount of interest to be paid on the 19th instalment will be the sum of $I_1, I_2, I_3, \ldots, I_{18}$

$$\therefore \quad I_1 + I_2 + ... + I_{18} = \left(\frac{900\times1\times5}{12\times100}\right) + \left(\frac{850\times1\times5}{12\times100}\right) + \dots + \left(\frac{50\times1\times5}{12\times100}\right)$$

$$= \frac{5}{1200}[900 + 850 + ... + 50]$$

$$= \frac{5}{1200}\left[\frac{18}{2}(900 + 50)\right] = 35.63$$

$\therefore$ Total interest to be paid = Rs.35.63

Example1.26: The sum to n terms of a certain series is given $2n^2 - 3n$. Show that the series is an A.P.

Solution: Given $S_n = 2n^2 - 3n$

But we know that, $S_n = t_1 + t_2 + t_3 + ... + t_{n-1} + t_n$ .... (1)

$S_{n-1} = t_1 + t_2 + t_3 + ... + t_{n-2} + t_{n-1}$ .... (2)

From (1) and (2), it is clear that, $S_{n-1} + t_n = S_n$

(i.e.,) $t_n = S_n - S_{n-1}$

$= (2n^2 - 3n) - [2(n-1)^2 - 3(n-1)]$

$= 2n^2 - 3n - [2n^2 - 4n + 2 - 3n + 3]$

$t_n = 4n - 5$

Putting $n = 1, 2, 3 ...$ we get,

$t_1 = 4(1) - 5 = -1$

$t_2 = 4(2) - 5 = 3$

$t_3 = 4(3) - 5 = 7$

Thus, the series $-1, 3, 7, ...$ form an A.P with common difference $d = 4$

Example1.27 : Solve : $1 + 6 + 11 + 16 + ... + x = 148$

Solution : The given series is an A.P. with $a = 1$ and $d = 6 - 1 = 5$.

Let x be the n^{th} term. $\therefore$ Given $S_n = 148$

$$\Rightarrow \quad \frac{n}{2}[2a + (n-1)d] = 148 \Rightarrow n[(2\times1) + (n-1)5] = 296$$

$$\Rightarrow \quad n(2 + 5n - 5) = 296 \Rightarrow 5n^2 - 3n - 296 = 0$$

$$\therefore \quad n = \frac{3\pm\sqrt{9+5920}}{10} = \frac{3\pm\sqrt{5929}}{10} = \frac{3\pm77}{10} \quad \therefore n = 8 \text{ or } \frac{-74}{10}$$

Rejecting $n = \frac{-74}{10}$ [as n cannot be negative], we get,

$n = 8 \quad \Rightarrow x = n^{th}$ term

$x = a + (n-1)\, d \quad \Rightarrow x = 1 + (8-1)\, 5 = 36$

Example1.28: If S_1, S_2, S_3 be the sum of n terms of three arithmetic series, the first term of each being 1 and the respective common differences are 1, 2, 3, prove that $S_1 + S_3 = 2S_2$

Proof : Let the first sequence be 1, 2, 3, 4, ... , n terms,

second sequence be 1, 3, 5, 7, ..., n terms and

third sequence be 1, 4, 7, 10, ..., n terms

$$S_n = \frac{n}{2}\,[2a + (n-1)\,d]$$

$$S_1 = \frac{n}{2}\,[2 + (n-1)\,1]$$

$$S_1 = \frac{n}{2}\,[\,n + 1] \quad \text{.................}\ (1)$$

$$S_2 = \frac{n}{2}\,[2 + (n-1)\,2] = \frac{n}{2}\,[2 + 2n - 2]$$

$$S_2 = n^2 \quad \text{..................}\ (2)$$

$$S_3 = \frac{n}{2}\,[2 + (n-1)\,3] = \frac{n}{2}\,[2 + 3n - 3]$$

$$S_3 = \frac{n}{2}\,[3n - 1] \quad \text{............}(3)$$

To prove $S_1 + S_3 = 2S_2$,

L.H.S. $S_1 + S_3 = \frac{n}{2}\,[\,n + 1] + \frac{n}{2}\,[3n - 1]$ from (1) & (3)

$$\Rightarrow \frac{n^2 + n + 3n^2 - n}{2} = 2n^2 = 2S_2 = \text{R.H.S} \qquad \therefore S_1 + S_3 = 2S_2$$

Example1.29: The interior angles of a polygon are in Arithmetic Progression.The smallest angle is 120^o and the common difference is 5^o. Find the number of sides of the polygon.

Solution : Here $a = 120^\circ$ and $d = 5^\circ$

Let S_n be the sum of the interior angles of a polygon of n sides.

$$\therefore S_n = \frac{n}{2}\,[2\,(120^\circ) + (n-1)\,5^\circ] = (n-2) \times 180^\circ$$

[$\because$ Sum of the interior angles of a polygon = (number of sides – 2) 180°]

$$\Rightarrow \quad \frac{n}{2}\,[240 + 5n - 5\,] = (n-2)180$$

$$\Rightarrow \quad 5n^2+235n = 360\,(n-2)$$

$$\div 5 \;\Rightarrow \quad n^2 - 25n + 144 = 0$$

$$\Rightarrow \quad n = 9 \text{ or } n = 16$$

when $n = 16$, n^{th} angle (i.e.,) $t_n = a + 15\,d = 120 + 15(5^\circ) = 195^\circ$

But the interior angle can not be more than 180°

$\therefore n = 16$ is rejected. Hence $n = 9$ only. (i.e.,) number of sides of the polygon is 9

EXERCISE 1.3

1. Find the sum of the following :

   (i) $3 + 1 - 1 - 3 - 5 - \ldots$ to 15 terms

   (ii) $100 + 97 + 94 + \ldots$ to 20 terms

   (iii) $6 + 5\frac{1}{4} + 4\frac{1}{2} + \ldots$ to 21 terms

   (iv) $0.50 + 0.52 + 0.54 + \ldots$ to 40 terms

   (v) $1 + \frac{1}{2} + 0 + \ldots$ to 20 terms

2. Find the sum to 20 terms of an A.P whose n^{th} term is $3n - 1$.
3. The sum to n terms of a certain series is given as $2n^2 + 3n$. Show that the series is an A.P.
4. Find the A.P in which the sum of any number of terms is always equal to three times the square of number of terms.
5. Find the sum of all numbers between 100 and 1000 which are divisible by 11.
6. Find the sum of all positive integers less than 298 which are multiples of 9.
7. Find the sum of all 3 digit numbers which leaves the remainder 1 when divided by 4.
8. How many terms of the series $1 + 6 + 11 + \ldots$ must be taken so that their sum is 970?
9. If $1 + 2 + 3 + \ldots + n = 666$, find '$n$'.
10. The first term of the A.P is 17. The number of terms is 21 and their sum is –63. Find the common difference and the middle term.

11. A man saved Rs.16500 in ten years. In each year after the first, he saved Rs. 100 more than he did in the preceding year. How much did he save in the first year?

12. 10 potatoes are planted in straight line on the ground. The distance between any two consecutive potatoes is 10 meters. How far must a person travel to bring them one by one to a basket placed 10 metres behind the first potato?

13. In an A.P the sum of first 10 terms is 175 and the sum of next 10 terms is 475. Find the A.P.

14. If the ratio of sum of m terms of an A.P to the sum of n terms is $m^2 : n^2$, show that the common difference is twice the first term and also prove that the ratio of the m^{th} term to the n^{th} term is $(2m-1) : (2n-1)$.

15. If S_1, S_2, S_3 be the sum to n, $2n$ and $3n$ terms respectively of an A.P, show that $S_3 = 3(S_2 - S_1)$.

1.4. GEOMETRIC PROGRESSION (G.P)

1.4.1. Geometric Sequence:

Consider the following sequences;

(i) 5, 10, 20, 40, ... (ii) 10, 5, $\frac{5}{2}$, ... (iii) $\frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \ldots$

These sequences are not in A.P. But they have a definite pattern.

From (i) we find, $\frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3} = \ldots = 2$

From (ii) we find , $\frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3} = \ldots = \frac{1}{2}$

From (iii) we find, $\frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3} = \ldots = 3$

It means each term of the sequence except the first term is obtained by multiplying the preceding term by a constant factor. Such a sequence is called a Geometric Sequence.

Geometric Sequence is also known as Geometric Progression.

Definition: A sequence of non-zero number is said to be a Geometric Progression (abbreviated as G.P) if the ratio of each term, except the first one, to its preceding term is always same (constant).

This constant ratio is called the common ratio of the G.P. The common ratio of the G.P is denoted by 'r'. The first term of the G.P is generally denoted by 'a'.

Note : In G.P, neither $a = 0$ nor $r = 0$.

Illustration : (i) The sequence 1, 2, 4, 8, 16, ... is a G.P with common ratio 2

$$\text{since } \frac{2}{1} = \frac{4}{2} = \frac{8}{4} = \ldots = 2$$

(ii) The sequence 27, 9, 3, 1, ... is a G.P with common ratio $\frac{1}{3}$

$$\text{since } \frac{9}{27} = \frac{3}{9} = \frac{1}{3} = \frac{\frac{1}{3}}{1} = \ldots = \frac{1}{3}$$

General Form of G.P

A Geometric Progression is a sequence of numbers in which the first term (a) is non-zero and each term except the first term is obtained by multiplying the term immediately preceding it with a fixed non-zero number (r).

The general form of a G.P is $a, ar, ar^2, \ldots$ with $a \neq 0$ and $r \neq 0$

General Term of G.P:

Let the G.P be $a, ar, ar^2, \ldots$

1st term $t_1 = a = ar^{1-1}$

2nd term $t_2 = ar = ar^{2-1}$

3rd term $t_3 = ar^2 = ar^{3-1}$

4th term $t_4 = ar^3 = ar^{4-1}$

Proceeding like this, we get the n^{th} term (or) general term of a G.P as $t_n = ar^{n-1}$

Example1.30: Find which of the following are G.P.

(a) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ (b) 45, 15, 5, ...

Solution : (a) In the above series $r = \frac{t_2}{t_1} = \frac{1}{3} \times \frac{2}{1} = \frac{2}{3}$

Also, $r = \frac{t_3}{t_2} = \frac{3}{4}$

Since their ratios are not equal, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is not a G.P.

(b) In the given series $r = \frac{t_2}{t_1} = \frac{15}{45} = \frac{1}{3}$

$$\text{Also, } r = \frac{t_3}{t_2} = \frac{5}{15} = \frac{1}{3}$$

Here the ratios are same. $\therefore$ 45, 15, 5, ... is a G.P.

Example1.31: Find the 6th term of the G.P $\frac{3}{16}, \frac{1}{8}, \frac{1}{12}, \cdots$

Solution : Here $a = \frac{3}{16}, \quad r = \frac{2}{3}$

$$\text{The 6th term} = t_6 = ar^5 = \left(\frac{3}{16}\right)\left(\frac{2}{3}\right)^5 = \frac{2}{81}$$

Example1.32: If the first term of a G.P is –5 and the common ratio is –2, find the G.P.

Solution : Given $a = -5, \quad r = -2$

General form of G.P is $a, ar, ar^2 \ldots$

$\therefore$ Required G.P is $-5, -5(-2), -5(-2)^2, \ldots$

(i.e.,) –5, 10, –20, 40, ...

Example1.33: Find the G.P whose 4th term is 8 and the 8th term is $\frac{128}{625}$.

Solution : Given $t_4 = 8, \quad t_8 = \frac{128}{625}$

(i.e.,) $t_4 = ar^3 = 8, \quad t_8 = ar^7 = \frac{128}{625}$

$$\frac{t_8}{t_4} \Rightarrow \frac{ar^7}{ar^3} = \frac{128}{625} \times \frac{1}{8}$$

$$\Rightarrow r^4 = \frac{16}{625} = \left(\frac{2}{5}\right)^4 \Rightarrow r = \pm\frac{2}{5}$$

If $r = \frac{2}{5}, \; a = ?$

We have $ar^3 = 8$

$$\Rightarrow a \times \left(\frac{2}{5}\right)^3 = 8 \qquad \Rightarrow a = \left[8 \times \frac{125}{8}\right] = 125$$

The required G.P is 125, 50, 20, ...

If $r = -\frac{2}{5}$, $a = ?$

We have $ar^3 = 8$

$$\Rightarrow a\left(-\frac{2}{5}\right)^3 = 8$$

$$\Rightarrow a = 8\left(-\frac{125}{8}\right) = -125$$

The required G.P is −125, 50, −20, ...

Example1.34 : Which term of the Progression 1, 2, 4, 8, ... is 512?

Solution : In the above G.P $a = 1$, $r = 2$

$\Rightarrow t_n = 512$ (i.e.,) $ar^{n-1} = 512$

$1(2)^{n-1} = 2^9 \quad \therefore n = 10$

Example1.35: Show that in a G.P the product of any two terms equidistant from the beginning and the end is equal to the product of the first and the last term.

Solution: Let the finite G.P be $a, ar, ar^2, ar^3, ..., ar^{n-1}$

Consider the p^{th} term from the beginning = ar^{p-1}

Consider the p^{th} term from the end = $ar^{n-1}\left(\frac{1}{r}\right)^{p-1}$

($\therefore$ Last term becomes the first term)

(p^{th} term from the beginning) (p^{th} term from the end)

$$= ar^{p-1}\, ar^{n-1}\left(\frac{1}{r}\right)^{p-1} = a.(ar^{n-1})$$

= (First term) (Last term)

Example1.36: The sum of the first two terms of a G.P is −8 and the sum of the first four terms is −80. Find the G.P.

Solution : Given the first two terms of a G.P = −8

(i.e.,) $a + ar = -8$

$\Rightarrow$ $a[1 + r] = -8$ ....(1)

Also given, the sum of the first four terms = – 80.

(i.e.,) $a + ar + ar^2 + ar^3 = -80$

$\Rightarrow$ $a[1+r] + ar^2[1+r] = -80$

$\Rightarrow$ $-8 + (-8)r^2 = -80$ [from (i)]

$\Rightarrow$ $r = \pm 3$

Substituting $r = 3$ in (1), we get, $a = -2$

When $a = -2$, $r = 3$

The G.P is −2, −6, −18, ...

When $r = -3$, we have $a = 4$

In this case, the G.P is 4, −12, 36, −108, ...

Example 1.37: Find three numbers in G.P such that their sum and product are respectively 14 and 64.

Solution : Let the three numbers in G.P be $\frac{a}{r}$, a, ar

Given, $\frac{a}{r} + a + ar = 14$ .... (1) and

$\frac{a}{r} \times a \times ar = 64 \quad \Rightarrow \quad a = 4$

Substituting $a = 4$ in (1), we get,

$$4r^2 - 10r + 4 = 0 \qquad \Rightarrow \qquad r = \frac{1}{2}, \ 2$$

When $a = 4$ and $r = \frac{1}{2}$, the required 3 numbers are 8, 4, 2.

When $a = 4$ and $r = 2$, the required 3 numbers are 2, 4, 8.

Example1.38: The first term of a G.P is 64 and the common ratio is r. Find 'r' if the average of the first and fourth terms is 140.

Solution: Let the G.P be $a, ar, ar^2, ar^3, \ldots$ Here $a = 64$

Given $\frac{a + ar^3}{2} = 140$

$\Rightarrow$ $\frac{a\left[1 + r^3\right]}{2} = 140$

$$\Rightarrow \frac{64\left[1+r^3\right]}{2} = 140$$

$$\Rightarrow \quad 1 + r^3 = \frac{140}{32}$$

$$\Rightarrow \quad r^3 = \frac{140-32}{32} = \frac{108}{32}$$

$$r^3 = \frac{27}{8} \quad \therefore r = \frac{3}{2} = 1\frac{1}{2}$$

The required common ratio 'r' $= 1\frac{1}{2}$

Example1.39: If p^{th}, q^{th}, r^{th} term of a G.P are x, y, z, show that $x^{q-r} y^{r-p} z^{p-q} = 1$.

Solution : Let 'a' be the first term and 'R' be the common ratio.

$x = a R^{p-1}$ ....(1)

$y = a R^{q-1}$ ....(2)

$z = a R^{r-1}$ ....(3)

Raising (1) to power $q-r$, (2) to power $r-p$ (3) to power $p-q$ and multiplying (1), (2) and (3), we get,

$x^{q-r} y^{r-p} z^{p-q}$

$= [a^{q-r} R^{(p-1)(q-r)}] [a^{r-p} R^{(q-1)(r-p)}] [a^{p-q} R^{(r-1)(p-q)}]$

$= a^{q-r+r-p+p-q} \cdot R^{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)}$

$= a^0 R^0 = 1$

$\therefore x^{q-r} y^{r-p} z^{p-q} = 1.$ Hence proved.

Example 1.40: If a, b, c are in G.P and $a^x = b^y = c^z$, show that $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P.

Solution : Given a, b, c are in G.P.

$\therefore b^2 = ac$ ....(1)

Let $a^x = b^y = c^z = k$

$\therefore a = k^{\frac{1}{x}}, \; b = k^{\frac{1}{y}}, \; c = k^{\frac{1}{z}}$ ....(2)

Substituting (2) in (1), we get,

$$\left[k^{\frac{1}{y}}\right]^2 = \left[k^{\frac{1}{x}}\right]\left[k^{\frac{1}{z}}\right]$$

$$\Rightarrow \quad \frac{2}{y} = \frac{1}{x} + \frac{1}{z}$$

$\therefore \ \frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P. Hence proved.

Example 1.41: Three numbers are in Arithmetic Progression and their sum is 15. If 1, 3, 9 are added to them respectively, they form a G.P. Find the numbers.

Solution : Let the three numbers in A.P be $a-d, \ a, \ a+d$.

Given $(a-d)+(a)+(a+d) = 15 \quad \Rightarrow \quad a = 5$

According to the problem $(a-d)+1, (a)+3, (a+d)+9$ form a G.P.

(i.e.,) $6-d, \ 8, \ 14-d$ [where $a=5$]

$$\Rightarrow \quad \frac{8}{6-d} = \frac{14+d}{8}$$

$$\Rightarrow \quad [14+d]\ [6-d] = 64$$

$$\Rightarrow \quad -d^2 - 8d + 84 = 64$$

$$\Rightarrow \quad d^2 + 8d - 20 = 0 \qquad \therefore d = -10, \ 2$$

When $a = 5, d = -10$, the required numbers are 15, 5, –5

When $a = 5, d = 2$, the required numbers are 3, 5, 7

Note : Verify that when 1, 3, 9 are added to the numbers respectively they form a G.P.

Example 1.42: Find three numbers *a, b, c* between 2 and 18 such that their sum is 25. The numbers 2, *a*, *b* are in A.P and the numbers *b, c*, 18 are in G.P.

Solution : The three given numbers are *a, b, c*

Given $a + b + c = 25$ .... (1)

Also given, 2, a, b are in A.P $\therefore 2a = b + 2$ .... (2)

$$\Rightarrow a = \frac{b+2}{2}$$

Also, $b,\ c$, 18 are in G.P $\therefore\ c^2 = 18b$ ....(3)

(1) $\Rightarrow$ $\left(\frac{b+2}{2}\right) + b + c = 25 \Rightarrow b + 2 + 2b + 2c = 50$

$\Rightarrow$ $3b + 2c = 48$ ....(4)

(4) × 6 $\Rightarrow$ $18b + 12c = 288 \Rightarrow$ $c^2 + 12c - 288 = 0$

$\Rightarrow$ $c = -24, 12$ [But 'c' cannot be negative since it lies between 2 & 18]

Substituting $c = 12$ in (4), we get, $3b + 24 = 48 \Rightarrow b = 8$

We have $a + b + c = 25 \Rightarrow a + 8 + 12 = 25 \Rightarrow a = 5$

$\therefore a = 5,\ b = 8,\ c = 12$ $\therefore$ The required three numbers are 5, 8, 12.

EXERCISE 1.4

1. Find the common ratio of the following G.Ps;

   (i) $\sqrt{5}, -1, \frac{1}{\sqrt{5}}, -\frac{1}{5}, \ldots$ (ii) 0.1, 0.04, 0.016, ...

   (iii) 17, –17, 17, –17, ... (iv) $\frac{5}{2}, \frac{25}{4}, \frac{125}{8}, \ldots$

2. Find which of the following are in G.P.?

   (i) $2, -2\sqrt{3}, 6, \ldots$ (ii) 4, 8, 12, ...

   (iii) $\frac{1}{\sqrt{2}}, -1, 2\sqrt{2}, 2, \ldots$ (iv) 256, 128, 64, ...

   (v) 5, 55, 555, ... (vi) $1^2, 2^2, 3^2, \ldots$

3. (i) Find the 9^{th} term of the G.P 6, –12, 24, ...

   (ii) In a G.P $t_5 = 72$, $t_3 = 18$, find t_7.

   (iii) Find the 5^{th} term of the G.P whose 3^{rd} term is $\frac{3}{8}$ and 7^{th} term is $\frac{3}{128}$.

   (iv) The 3^{rd} term of the G.P is $\frac{2}{3}$ and the 6^{th} term is $\frac{2}{81}$. Find the G.P.

4. If 729, x, 9 are in G.P, find x .

5. The sum of the first 2 terms of a G.P is –1 and the sum of the first four terms is –5. Find the G.P.

6. If a, b, c, d are in G.P, prove that $ax^2 + c$ divides $ax^3 + bx^2 + cx + d$

7. Find the 3 numbers of G.P such that their respective sum and product are

(i) 6, – 64 (ii) $4\frac{1}{3}$, 1 (iii) –105, – 8000

8. Find 3 numbers in G.P such that their sum is $17\frac{1}{3}$ and the product of their reciprocals is $\frac{1}{64}$.

9. Find 3 numbers in G.P such that their sum is – 2 and sum of their squares is 12.

10. The sum of 3 terms of a G.P. is 7 and the sum of their squares is 21. Find the three numbers.

11. Find 4 numbers in G.P whose sum is 85 and product is 4096.

12. Find 5 numbers in G.P such that their product is 32 and the product of the last two numbers is 108.

13. Find 3 numbers in G.P whose sum is 52 and the sum of their product in pairs is 624.

14. Three numbers whose sum is 15 are in A.P. If 1, 4 and 19 are added to them respectively, the results are in G.P. Find the numbers.

15. The second term of a G.P is *'b'* and the common ratio is *'r'*. Write down the value of *'b'* if the product of the first three terms is 64.

1.5. GEOMETRIC SERIES

1.5.1. Sum of the first *'n'* terms of a G.P:

There is an interesting story connected with the sum of a Geometric Progression. It is said that the King of Persia was so pleased with the inventor of the game of chess (believed to be of Indian origin) that he offered to give him any reward. The inventor wanted one grain of wheat to be placed on the first square of the chessboard, two grains on the second square, four on the third, eight on the fourth and so on. The demand of the inventor appeared to be very modest and the King thought that it was not an enough reward. Is the reward petty?

We see that the grains to be placed in the various squares are the terms of the sequence 1, 2, 2^2, 2^3, ...,2^{63}.

Sum of the grains = $S_{64} = 1 + 2 + 2^2 + ... + 2^{63}$ ...... (1)

Consider in reverse order

$$S_{64} = 2^{63} + 2^{62} + ...+ 2^2 + 2 + 1$$

$$2\,S_{64} = 2^{64} + 2^{63} + ...+ 2^3 + 2^2 + 2 \qquad ......\ (2)$$

Subtract (1) from (2), we get,

$$S_{64} = 2^{64} - 1$$

Can you imagine the quantum of grains? It is said that the amount of wheat the inventor had asked would have covered our large country with a layer of wheat to almost 4 cm deep.

Let S_n denote the sum of n terms of the G.P with first term 'a' and common ratio 'r'. Then,

$$S_n = a + ar + ar^2 + ... + ar^{n-2} + ar^{n-1} \quad ..... (1)$$

Multiplying both sides by 'r' we get,

$$rS_n = ar + ar^2 + ar^3 + ... + ar^{n-1} + ar^n \quad ..... (2)$$

Subtract (2) from (1), we get,

$$S_n - rS_n = a - ar^n$$

$$(1-r)\, S_n = a\,(1-r^n)$$

$$S_n = \frac{a\left(1-r^n\right)}{1-r}$$

The above formula can be considered in the following ways;

$$S_n = \begin{cases} \dfrac{a\left(1-r^n\right)}{1-r} & if \quad r<1 \\ \dfrac{a\left(r^n-1\right)}{r-1} & if \quad r>1 \end{cases}$$

But the above formulae do not hold for $r = 1$

If $r = 1$, the G.P reduces to $a, a, a,$ ... upto n terms.

$\therefore\ S_n = a + a + a + a +$ ... upto n terms.

$$S_n = na$$

1.5.2. Sum to Infinite Terms of G.P:

If $-1 < r < 1$, say $r = \frac{1}{2}$, then $r^2 = \frac{1}{4}$, $r^3 = \frac{1}{8}$, ... $r^n = \frac{1}{2^n}$ is very small and r^n approaches to 0 when n is very large.

We know that sum of n terms of this G.P is S_n = a $\left(\dfrac{1-r^n}{1-r}\right)$

$$S_n = \frac{a}{1-r} - \frac{ar^n}{1-r} \qquad \text{............(1)}$$

Since $|r| < 1$, r^n decreases as 'n' increases

$$\therefore \frac{ar^n}{1-r} \text{ approaches to } 0$$

$$\therefore \text{(1) becomes } S_\infty = \frac{a}{1-r} \text{ if } |r| < 1$$

Note : If $|r| \geq 1$, then the sum of an infinite G.P approaches to infinity.

Example 1.43: Find the sum of the series 1 + 3 + 9 + ... to 10 terms.

Solution: Here $a = 1$, $r = 3$

We have $$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{1\left(3^{10} - 1\right)}{2} = 29524$$

Example 1.44: Find sum of the series 243 + 324 + 432 + ... to n terms.

Solution : Here $a = 243$, $r = \dfrac{324}{243} = \dfrac{4}{3}$

Here $r > 1$ $$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{243\left[\left(\frac{4}{3}\right)^n - 1\right]}{\left[\frac{4}{3} - 1\right]} = \frac{729\left[4^n - 3^n\right]}{3^n}$$

$$= \frac{3^6\left[4^n - 3^n\right]}{3^n}$$

$$\therefore S_n = 3^{6-n}\left[4^n - 3^n\right]$$

Example 1.45: Find the sum to infinity of the G.P 10, –9, 8.1, ...

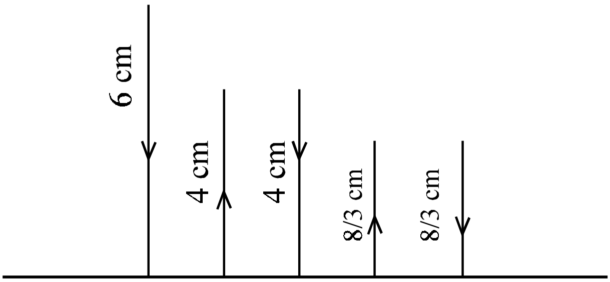
Solution : Here $a = 10$, $r = \frac{-9}{10}$ and $|r| = \left|\frac{-9}{10}\right| = \frac{9}{10} < 1$

$\therefore$ Sum to infinity is defined.

$$S_\infty = \frac{a}{1-r} = \frac{10}{1-\left(\frac{-9}{10}\right)} = \frac{10}{1+\frac{9}{10}} = \frac{100}{19}$$

Example 1.46: A ball is dropped from a height of 6m and on each bounce it rebounces to $\frac{2}{3}$ of its previous height. (i) What is the total length of the downward paths? (ii) What is the total length of the upward paths? (iii) How far does the ball travel till it stops bouncing?

Solution :



(i) Distance covered in the downward path = $6 + 4 + \frac{8}{3} + \frac{16}{9} + ...$

[it is an infinite geometric series with $a = 6$, $r = \frac{2}{3}$]

$$= \frac{6}{\left[1-\frac{2}{3}\right]} = 18 \text{ m}$$

(ii) Distance covered in the upward path = $4 + \frac{8}{3} + \frac{16}{9} + ...$

[it is an infinite geometric series with $a = 4$, $r = \frac{2}{3}$]

$$= \frac{4}{\left[1-\frac{2}{3}\right]} = 12 \text{ m}$$

(iii) Total distance covered = 18 + 12 = 30 m

Example 1.47: Find the sum to n terms of the series 7+ 77 + 777 + ... to n terms.

Solution : Given S_n = 7 + 77 + 777 + ... to n terms

$$= 7\,(1 + 11 + 111 + \ldots \text{ to } n \text{ terms})$$

$$= \frac{7}{9}\,(9 + 99 + 999 + \ldots \text{ to } n \text{ terms})$$

$$= \frac{7}{9}\,[(10-1) + (100-1) + \ldots \text{ to } n \text{ terms})]$$

$$= \frac{7}{9}\,[(10 + 100 + \ldots \text{ to } n \text{ terms}) - (1+1+\ldots \text{ to } n \text{ terms})]$$

$$= \frac{7}{9}\,[(10^1 + 10^2 + \ldots \text{ to } n \text{ terms}) - n]$$

$$= \frac{7}{9}\left[\frac{10\left(10^n - 1\right)}{9} - n\right] = \frac{70}{81}\left(10^n - 1\right) - \frac{7n}{9}$$

Example 1.48: Find the least value of n for which the sum $1 + 3 + 3^2 + \ldots$ to n terms is greater than 7000.

Solution : Given that sum to n terms = $1 + 3 + 3^2 + \ldots$ to n terms

(i.e.,) $S_n = 1 + 3 + 3^2 + \ldots + 3^{n-1}$

$$= \frac{1\left(3^n - 1\right)}{3-1} = \frac{3^n - 1}{2}$$

Also given, sum to n terms > 7000 $\quad\therefore \dfrac{3^n - 1}{2} > 7000$

$$\Rightarrow 3^n > 14001$$

To find n take log on both sides, we get,

$$n \log 3 > \log(14001)$$

$$n > \frac{\log(14001)}{\log 3} = \frac{4.1461}{0.4771} = 8.69$$

$$\therefore n > 8.69$$

Hence, the least value of n is 9.

Another method :

We have $3^2 = 9$; $3^4 = 81$; $3^6 = 729$; $3^8 = 6561$; $3^9 = 19683$

The least value of n is 9.

Example 1.49: The sum an infinite series in G.P is 57 and the sum of their cubes is 9747. Find the series.

Solution : Let the infinite series in G.P be $a + ar + ar^2 + \ldots \infty$

Given $S_n = \dfrac{a}{1-r} = 57$ ....(1)

Also given, the sum of their cubes is 9747

(i.e.,) $a^3 + a^3r^3 + a^3r^6 + \ldots \infty = 9747$

Sum to infinite terms of the above series is equal to $\dfrac{a^3}{1-r^3} = 9747$ ......(2)

Cubing the first equation, we get, $\dfrac{a^3}{(1-r)^3} = 57^3$ ....(3)

$$(3) \div (2) \quad \Rightarrow \quad \frac{a^3}{(1-r)^3} \times \frac{1-r^3}{a^3} = \frac{57^3}{9747}$$

$$\Rightarrow \quad \frac{(1-r)\left(1+r+r^2\right)}{(1-r)(1-r)(1-r)} = 19$$

$$\Rightarrow \quad 18r^2 - 39r + 18 = 0$$

$$\Rightarrow \quad r = \frac{3}{2} \text{ (or) } \frac{2}{3}$$

Sum of an infinite series in G.P exists only if $r < 1$

$\therefore$ The value of $r = \dfrac{2}{3}$

$$(1) \quad \Rightarrow \quad \frac{a}{1-\frac{2}{3}} = 57 \qquad \Rightarrow \quad a = 19.$$

Hence the required series is $19 + \dfrac{38}{3} + \dfrac{76}{9} + \ldots$ to ∞ .

Example 1.50: Express $0.1\overline{23}$ as a fraction.

Solution : $0.1\overline{23}$ $= 0.1232323 \ldots$

$$= 0.1 + 0.023 + 0.00023 + \ldots$$

$$= \frac{1}{10} + \frac{23}{1000} + \frac{23}{100000} + \ldots$$

$$= \frac{1}{10} + \frac{23}{10^3} + \frac{23}{10^5} + \ldots$$

$$= \frac{1}{10} + \frac{\left[\frac{23}{10^3}\right]}{\left[1 - \frac{1}{10^2}\right]}$$

$$= \frac{1}{10} + \frac{23}{990} = \frac{122}{990}$$

Example 1.51: Express $0.\overline{325}$ as a fraction.

Solution : 0.325 $= 0.325325325 \ldots$

$$= \frac{325}{1000} + \frac{325}{1000000} + \ldots$$

$$= \frac{325}{10^3} + \frac{325}{10^6} + \ldots$$

$$= \frac{\left[\frac{325}{10^3}\right]}{\left[1 - \frac{1}{10^3}\right]} = \frac{325}{999}$$

Example 1.52: In an infinite G.P each term is equal to three times the sum of all the terms that follow it and the sum of the first two terms is 15. Find the sum of the series to infinity.

Solution : Let '*a*' be the first term and '*r*' be the common ratio of the given series.

Then the series be $a + ar + ar^2 + \ldots$ to ∞

According to the problem, n^{th} term $= 3\,[ar^n + ar^{n+1} + ...\infty]$

$$ar^{n-1} = 3\left[\frac{ar^n}{1-r}\right]$$

$$\Rightarrow \quad 1 - r = 3r$$

$$\Rightarrow \quad r = \frac{1}{4}$$

Also given, $a + ar = 15$

$$\Rightarrow \quad a\,(1 + r) = 15$$

$$\Rightarrow \quad a\left(1+\frac{1}{4}\right) = 15 \quad \Rightarrow \quad a = 12$$

$$S_\infty = \frac{a}{1-r} = \frac{12}{1-\frac{1}{4}} = 16$$

Example 1.53: If $S_1,\ S_2,\ S_3$ be the sums of n, $2n$, $3n$ terms respectively of a G.P, prove that $S_1\left(S_3 - S_2\right) = \left(S_2 - S_1\right)^2$.

Proof: Here $S_1 = \dfrac{a\left(r^n - 1\right)}{r-1}$, $S_2 = \dfrac{a\left(r^{2n}-1\right)}{r-1}$ and $S_3 = \dfrac{a\left(r^{3n}-1\right)}{r-1}$

To prove $S_1\left(S_3 - S_2\right) = \left(S_2 - S_1\right)^2$

L.H.S. $= S_1\left(S_3 - S_2\right)$

Now,
$$S_3 - S_2 = \left[\frac{a\left(r^{3n}-1\right)}{r-1}\right] - \left[\frac{a\left(r^{2n}-1\right)}{r-1}\right]$$

$$= \frac{a}{r-1}\left[r^{3n} - r^{2n}\right]$$

$$\therefore\ S_1(S_3 - S_2) = \frac{a\left(r^n-1\right)}{(r-1)}\left[\frac{a\,r^{2n}\left(r^n-1\right)}{(r-1)}\right] = \frac{a^2 r^{2n}\left(r^n-1\right)^2}{(r-1)^2} \quad ....(1)$$

R.H.S.= $(S_2 - S_1)^2$

Now, $S_2 - S_1 = \left[\dfrac{a\left(r^{2n}-1\right)}{r-1}\right] - \left[\dfrac{a\left(r^{n}-1\right)}{r-1}\right]$

$$= \frac{a}{r-1}\left[r^{2n} - r^n\right] = \frac{a\,r^n}{(r-1)}\left(r^n - 1\right)$$

$$\therefore\ (S_2 - S_1)^2 = \frac{a^2 r^{2n}\left(r^n-1\right)^2}{(r-1)^2} \qquad \text{...............(2)}$$

From (1) and (2), L.H.S. = R.H.S.

Example 1.54 : Find the value of $\sqrt[3]{25\sqrt[3]{25\sqrt[3]{25...}}}$.

Solution : $\sqrt[3]{25\sqrt[3]{25\sqrt[3]{25...}}} = 25^{\frac{1}{3}} \times 25^{\frac{1}{9}} \times 25^{\frac{1}{27}} \times ...\infty$

$$= 25^{\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+...\infty\right)}$$

Let us find $\dfrac{1}{3}+\dfrac{1}{9}+\dfrac{1}{27}+ ...\infty$

$$\therefore S_\infty = \frac{\frac{1}{3}}{\left[1-\frac{1}{3}\right]} = \frac{1}{2}$$

$$25^{\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+...\infty\right)} = 25^{\frac{1}{2}} = \sqrt{25} = 5.$$

EXERCISE 1.5

1. Find the sum of the following G.Ps;

   (i) 3 + 9 + 27 + ... to 6 terms. (ii) 1 + 0.2 + 0.04 + ... to 8 terms.

   (iii) $\sqrt{2} + 2 + 2\sqrt{2}$ + ... to 8 terms (iv) $1 + \dfrac{1}{2} + \dfrac{1}{4}$ + ... to 7 terms.

2. Find the sum to infinity of the following :

(i) 4, 2, 1, ... (ii) $\frac{3}{16}, \frac{1}{8}, \frac{1}{12}, \frac{1}{18}, ...$

(iii) –2, 10, –50, 250, ... (iv) 0.9, 0.18, 0.36, ...

3. Find the sum to '*n*' terms of the series.

(i) $\frac{1}{3} + \left(-\frac{1}{9}\right) + \frac{1}{27} + \left(-\frac{1}{81}\right) + ...$ (ii) 9 + 99 + 999 + ...

(iii) 6 + 66 + 666 + ... (iv) 0.9 + 0.99 + 0.999 + ...

(v) 0.4 + 0.94 + 0.994 + ... (vi) 11 + 103 + 1005 + ...

4. Find the sum to infinity of the G.P.

(i) $4, \frac{4}{3}, \frac{4}{9}, ...$ (ii) $7, -1, \frac{1}{7}, ...$ (iii) $-\frac{5}{4}, \frac{5}{16}, \frac{-5}{64}, ...$

5. A ball is dropped from a height of 100 m. At each bounce it raises half the height it raised with the previous bounce. Find the distance covered by the ball till it comes to rest.

6. The sum to *n* terms of a G.P is 255, the last term is 128 and the common ratio is 2. Find '*n*'.

7. A G.P consists of 2*n* terms. If the sum of the terms occupying the odd places is S_1 and that of the terms in even places is S_2, prove that the common ratio of the progression is $\frac{S_2}{S_1}$.

8. If $S_n = 1 + 4 + 4^2 + ...$ to *n* terms, find the least value of '*n*' such that $S_n > 1500$.

9. The sum of an infinite G.P is 24 and its first term is 8. Find the G.P.

10. The sum of infinite terms in a G.P is 2 and the sum of their squares is $\frac{4}{3}$. Find the series.

11. If '*S*' be the sum, '*P*' the product and '*R*' the sum of the reciprocals of *n* terms in a G.P, prove that $P^2 = \left(\frac{S}{R}\right)^n$

12. Express the following into a fraction (i) $0.5\overline{73}$ (ii) $0.\overline{268}$.

13. The midpoints of 16 by 12 rectangle have been connected to form a rhombus. Then, midpoints of the rhombus are connected to form a rectangle, and so on.
(i) List the perimeters of the first 10 rectangles formed (including the largest rectangle)

(ii) What is the sum of the perimeters of the first 10 rectangles formed?

14. A hot air ball rises 80 meters in the first minute of flight. If in each succeeding minute the ballon rises only 90% as far as in the previous minute, what will be its maximum altitude?

15. Prove that the numbers 49, 4489, 444889, ... obtained by inserting 48 into the middle of the preceding number are squares of integers.

16. If $P = \sqrt[3]{3\times\sqrt[3]{3^2\times\sqrt[3]{3^3\times\ldots\infty}}}$, then find $\sqrt[3]{p}$.

17. If $x = a + \dfrac{a}{r} + \dfrac{a}{r^2} + \ldots\ \infty$; $y = b - \dfrac{b}{r} + \dfrac{b}{r^2} - \ldots\infty$; $z = c + \dfrac{c}{r^2} + \dfrac{c}{r^4} + \ldots\ \infty$,

show that $\dfrac{xy}{z} = \dfrac{ab}{c}$.

18. If $S_1, S_2, S_3, \ldots, S_p$ denote the sums of infinite G.Ps whose first terms are

1, 2, 3, ... , p respectively and whose common ratios are $\dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \ldots, \dfrac{1}{(p+1)}$

respectively, show that $S_1 + S_2 + S_3 + \ldots + S_p = \dfrac{p(p+3)}{2}$.

1.6 SPECIAL SERIES - $\sum n$, $\sum n^2$, $\sum n^3$

1.6.1. The $\sum$ - Notation:

The symbol $\sum$ (called Sigma) indicates a summation to be carried out.

For example, 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 is written as $\sum_{n=1}^{n=10} n$

The first value of n is 1 (i.e., n = 1). Then the consecutive integral values of n has to be taken. The last value of n is 10 (i.e., n = 10).

Remark: $\sum n$ is the simplified way of writing $\sum_{k=1}^{n} k$

1.6.2. Sum of the First '*n*' Natural Numbers:

$$\sum n = 1 + 2 + 3 + \dots + n.$$

Here $1 + 2 + 3 + \dots + n$ is an arithmetic series with $a = 1$, $d = 1$

$$\therefore \sum n = S_n = \frac{n}{2}[2a + (n-1)d]$$

$$= \frac{n}{2}[2 + (n-1)(1)] = \frac{n(n+1)}{2}$$

$$\therefore \sum n = \frac{n(n+1)}{2}$$

1.6.3. Sum of the Squares of the First '*n*' Natural Numbers:

$$\sum n^2 = 1^2 + 2^2 + 3^2 + \dots + n^2.$$

Let us consider the identity, $x^3 - (x-1)^3 = 3x^2 - 3x + 1$

Putting $x = 1, 2, 3, \dots n$, we get,

$$1^3 - 0^3 = 3(1)^2 - 3(1) + 1$$
$$2^3 - 1^3 = 3(2)^2 - 3(2) + 1$$
$$3^3 - 2^3 = 3(3)^2 - 3(3) + 1$$
$$4^3 - 3^3 = 3(4)^2 - 3(4) + 1$$
$$\dots \quad \dots \quad \dots \dots \dots \dots \dots \dots \dots \dots$$
$$\dots \quad \dots \quad \dots \dots \dots \dots \dots \dots \dots \dots$$
$$n^3 - (n-1)^3 = 3n^2 - 3n + 1$$

Adding columnwise, we get, $n^3 = 3(1^2 + 2^2 + 3^2 + \dots + n^2) - 3(1 + 2 + \dots + n) + (1 + 1 + 1 + 1 + \dots n \text{ terms})$

$$= 3(1^2 + 2^2 + 3^2 + \dots + n^2) - 3\,\frac{n(n+1)}{2} + n$$

$$\therefore 3(1^2 + 2^2 + 3^2 + \dots + n^2) = n^3 + 3\,\frac{n(n+1)}{2} - n$$

$$= \frac{2n^3 + 3n^2 + 3n - 2n}{2}$$

$$= \frac{2n^3 + 3n^2 + n}{2}$$

$$= \frac{n(2n^2 + 3n + 1)}{2}$$

$$\therefore \sum n^2 = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

1.6.4. Sum of the Cubes of the First '*n*' Natural Numbers:

$$\sum n^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3$$

Let us consider the following pattern

$$1^3 = 1 = 1^2$$

$$1^3 + 2^3 = 9 = (1 + 2)^2$$

$$1^3 + 2^3 + 3^3 = 36 = (1 + 2 + 3)^2$$

$$1^3 + 2^3 + 3^3 + 4^3 = 100 = (1 + 2 + 3 + 4)^2$$

Proceeding like this $1^3+2^3+3^3+ \ldots + n^3 = (1 + 2 + 3 + 4 \ldots + n)^2$

$$\sum n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Example 1.55: Find the sum of first 75 natural numbers.

Solution: $\sum_{1}^{75} n = 1 + 2 + 3 + 4 + . . . + 75 .$

$$= \frac{75(75+1)}{2} = \frac{75 \times 76}{2} = 2850$$

Example 1.56: Find the sum of 5 + 10 +15 + ... + 250.

Solution: $5 + 10 + 15 + ... + 250 = 5 [1 + 2 + 3 + ... + 50]$

$$= 5\left[\frac{50(50+1)}{2}\right] = 6375$$

Example 1.57: Find the sum of 15 + 16 + ... + 80.

Solution: $15 + 16 + ... + 80 = (1 + 2 + ... + 80) - (1 + 2 + ...+ 14)$

$$= \left[\frac{80 \times 81}{2}\right] - \left[\frac{14 \times 15}{2}\right] = 3135$$

Example 1.58: Find 5 + 7 + 9 + ... + 99, using the formula $\sum n$

Solution: 5 + 7 + 9 + ... + 99

$$= (1 + 2 + 3 + ...+ 99) - (2 + 4 + ... + 98) - (1+3)$$

$$= \left[\frac{99\times100}{2}\right] - 2\left[\frac{49\times50}{2}\right] - 4 = 2496$$

Example 1.59: Evaluate : 1 + 4 + 9 + ... + 1600.

Solution: $1 + 4 + 9 + ... + 1600 = 1^2 + 2^2 + 3^2 + ... + 40^2$

$$\therefore \sum n^2 = \frac{40(40+1)(2\times40+1)}{6}$$

$$= \left[\frac{40\times41\times81}{6}\right] = 22140$$

(i.e.,) $1^2 + 2^2 + 3^2 + ... + 40^2 = 22140$

Example 1.60: Find $21^2 + 22^2 + ... + 35^2$

Solution: $21^2 + 22^2 + ... + 35^2 = (1^2 + 2^2 + ... + 35^2) - (1^2+2^2+ ... +20^2)$

$$= \frac{35(35+1)(2\times35+1)}{6} - \frac{20(20+1)(2\times20+1)}{6}$$

$$= \frac{35\times36\times71}{6} - \frac{20\times21\times41}{6} = 12040$$

Example 1.61: Find $1^2 + 3^2 + 5^2 + ...+ 41^2$

Solution: $1^2+3^2+5^2+ ... +41^2 = (1^2 + 2^2 + ... + 41^2) - (2^2+4^2+ ... +40^2)$

$$= (1^2 + 2^2 + ... + 41^2) - (2\times1)^2+(2\times2)^2+ ... +(2\times20)^2$$

$$= (1^2 + 2^2+ ... + 41^2) - 2^2(1^2 + 2^2 + ... + 20^2)$$

$$= \left[\frac{41\times42\times83}{6}\right] - 4\left[\frac{20\times21\times41}{6}\right] = 23821 - 11480 = 12341$$

Example 1.62: Find $1^3+2^3+ ... + 15^3$

Solution: Given that $1^3+2^3+ ... + 15^3 = \left[\frac{15\times16}{2}\right]^2 = (120)^2 = 14400$

Example 1.63 : Evaluate: $11^3+12^3+ \ldots + k^3$ where $k = 50$

Solution: Given that $11^3 + 12^3 + \ldots + k^3 = 11^3 + 12^3 + \ldots + 50^3$

$$= (1^3 + 2^3 + \ldots + 50^3) - (1^3 + 2^3 + \ldots + 10^3)$$

$$= \left[\frac{50\times 51}{2}\right]^2 - \left[\frac{10\times 11}{2}\right]^2$$

$$= 1625625 - 3025 = 1622600$$

EXERCISE 1.6

1. Evaluate the following :

   (i) $1 + 2 + \ldots + 90$ (ii) $3 + 6 + \ldots + 210$

   (iii) $26 + 27 + \ldots + 65$ (iv) $1 + 3 + 5 + \ldots + 71$

   (v) $15 + 17 + \ldots + 45$ (vi) $1 + 4 + 9 + \ldots + 225$

   (vii) $10^2 + 11^2 + \ldots + 30^2$ (viii) $1^2 + 2^2 + \ldots + 11^2$

   (ix) $12^3 + 13^3 + \ldots + 30^3$ (x) $1 + 8 + 27 + \ldots + 125000$

2. Some cubes of sides 5 cm, 6 cm, 7 cm, ... , 20 cm are arranged in a box such that there is no space left in the box. Find the volume of the box.

3. The sum of the cubes of first *n* natural numbers is 44100. Find *n*.

4. If $1 + 2 + 3 + \ldots + k = 325$, find $1^3 + 2^3 + 3^3 + \ldots + k^3$

5. If $1^3 + 2^3 + 3^3 + \ldots + k^3 = 38025$, find $1 + 2 + 3 + \ldots + k$

6. How many terms of the series $1^3 + 2^3 + 3^3 + \ldots$ should be taken to get 25502500 ?

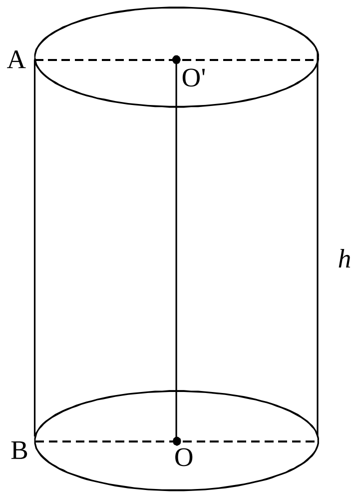
——————

2. MENSURATION

2.1 A RIGHT CIRCULAR CYLINDER

A right circular cylinder is a solid generated by the revolution of a rectangle about one of its sides.

Consider a rectangle OO' AB which revolves about its side OO' and completes one full round to arrive at its initial position. The revolution generates a right circular cylinder as shown in the figure.



A right circular cylinder has two plane ends. Each plane end is circular in shape and the two plane ends are parallel. Each of these two ends is called the base of the cylinder. The line segment joining the centres of the two circular ends is called the axis of the cylinder. The axis is always perpendicular to the bases of a right circular cylinder. The length of the axis of the cylinder is called its height and the radius of the circular bases is known as its radius. The curved surface joining the two bases of a right circular cylinder is called its lateral surface.

In the above figure, OO' is the axis of the cylinder, O'A = OB is its radius and OO' is the height.

For a right circular cylinder of radius r and height h. We have,

(1) Base area of the cylinder = πr^2 sq.units.

(2) Volume = Base area × height = $\pi r^2 \times h = \pi r^2 h$ cubic units.

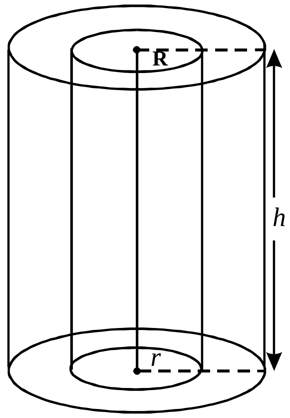
(3) Curved surface area = Circumference of the base × height
(Lateral surface area)
Curved surface area = $2 \pi r h$ sq. units.

(4) Total surface area = Curved surface area + 2 base areas

$= 2\pi rh + 2\pi r^2 = 2\pi r\,(h + r)$

Total surface area $= 2\,\pi r\,(h + r)$ sq.units.

2.2 HOLLOW CYLINDER



A solid bounded by two coaxial cylinders of the same height and different radii is called a hollow cylinder. If R is external and r is internal radii of a hollow cylinder of height h then

(1) Area of each end
$= \pi\,(R^2 - r^2)$ sq.units

(2) Curved surface area
(Lateral suface area)
= External surface area
+ Internal surface area

$= 2\pi Rh + 2\pi rh = 2\pi h\,(R + r)$ sq.units.

(3) Total surface area = Curved surface area + 2 Area of base rings

$= 2\pi h\,(R + r) + 2\pi\,(R + r)\,(R - r)$

$= 2\pi\,(R + r)\,(R - r + h)$

Total surface area $= 2\pi\,(R + r)\,(R - r + h)$ sq.units.

(4) Volume of the material = Exterior volume – Interior volume

$= \pi R^2 h - \pi r^2 h$

$= \pi h\,(R^2 - r^2)$

$= \pi h\,(R + r)\,(R - r)$

Volume of the material $= \pi h\,(R + r)\,(R - r)$ cu.units.

Example 2.1 **:** A cylinder has radius 7 cm and height 10 cm find (i) Volume, (ii) Curved surface area (ii) Total surface area of the cylinder.

Solution : Radius (r) = 7 cm

Height (h) = 10 cm

Volume of the cylinder $= \pi r^2 h$ cu.units

$= \frac{22}{7} \times 7 \times 7 \times 10 = 1540$ cu.cm.

Volume of the cylinder = 1540 cu.cm

Curved surface area = $2\pi rh$ sq.units

$$= 2 \times \frac{22}{7} \times 7 \times 10 = 440 \text{ sq.cm.}$$

Curved surface area = 440 sq.cm.

Total surface area = $2\pi r\,(h + r)$ sq.units.

$$= 2 \times \frac{22}{7} \times 7\,(10 + 7)$$

$$= 2 \times \frac{22}{7} \times 7 \times 17 = 748 \text{ sq.cm.}$$

Total surface area = 748 sq.cm.

Example 2.2 : The circumference of the base of a cylinder is 25.45 cm and its height is 15 cm. Find the curved surface area.

Solution : Circumference of the base = 25.45 cm, height = 15 cm

Curved surface area = circumference of the base × height

= 25.45 × 15 = 381.75 sq.cm.

Curved surface area = 381.75 sq.cm.

Example 2.3 : 12 cylindrical pillars of a building have to be cleaned. If the diameter of each pillar is 42 cm and the height of each pillar is 5 m, what will be the cost of cleaning these at the rate of Rs. 5. per sq.m.

Solution : Diameter of pillar = 42 cm

Radius (r) $= \frac{42}{2} = 21 \text{ cm} = \frac{21}{100}$ m

Height (h) = 5 m

Curved area of pillar = $2\pi\, rh$ sq.units

$$= 2 \times \frac{22}{7} \times \frac{21}{100} \times 5 = 6.6 \text{ sq.m.}$$

Total area of 12 pillars = 12 × 6.6 = 79.2 sq.m.

Cost of cleaning for 1 sq.m = Rs. 5

Cost of cleaning for 79.2 sq.m = 79.2 × 5 = 396

Cost of cleaning of 12 pillars = Rs. 396.

Example 2.4 : Find the radius of the cylinder if the area of its curved surface is 352 cm^2 and its height 16 cm.

Solution :

$$\text{Height} = 16 \text{ cm}$$

$$\text{Curved surface area} = 352 \text{ cm}^2$$

$$2\pi rh = 352$$

$$2 \times \frac{22}{7} \times r \times 16 = 352$$

$$r = 352 \times \frac{1}{2} \times \frac{7}{22} \times \frac{1}{16}$$

$$= \frac{7}{2} = 3.5 \text{ cm}$$

$$\text{Radius of cylinder} = 3.5 \text{ cm.}$$

Example 2.5 : The volume of a cylinder is 448 π cu.cm and height 7 cm. Find its curved surface area.

Solution :

$$\text{Height} = 7 \text{ cm.}$$

$$\text{Volume of the cylinder} = 448\,\pi \text{ cu.cm}$$

$$\pi r^2 h = 448\,\pi$$

$$r^2 = \frac{448}{7} = 64$$

$$r = \sqrt{64} = 8 \text{ cm}$$

$$\text{Curved surface area} = 2\pi rh \text{ sq.units.}$$

$$= 2 \times \pi \times 8 \times 7$$

$$= 112\,\pi \text{ sq.cm.}$$

$$= 112 \times \frac{22}{7} = 352 \text{ sq.cm.}$$

$$\text{Curved surface area} = 352 \text{ sq.cm.}$$

Example 2.6 : A solid cylinder has a total surface area of 231 cm^2. Its curved surface area is $\frac{2}{3}$ of the total surface area. Find the volume of the cylinder.

Solution :

$$\text{Total surface area} = 231 \text{ cm}^2$$

$$\text{Curved surface area} = \frac{2}{3} \text{ of T.S.A}$$

$$2\pi rh = \frac{2}{3} \times 231 = 154 \text{ cm}^2.$$

$$\text{Total surface area} = 231 \text{ cm}^2$$

$$2\pi r(h + r) = 231$$

$$2\pi rh + 2\pi r^2 = 231$$

$$154 + 2\pi r^2 = 231$$

$$2\pi r^2 = 231 - 154$$

$$2 \times \frac{22}{7} \times r^2 = 77$$

$$r^2 = 77 \times \frac{1}{2} \times \frac{7}{22} = \frac{49}{4}$$

$$r = \frac{7}{2}$$

$$r = 3.5 \text{ cm.}$$

$$\text{Curved surface area} = 154 \text{ cm}^2$$

$$2\pi rh = 154$$

$$2 \times \frac{22}{7} \times \frac{7}{2} \times h = 154$$

$$h = \frac{154}{22} = 7 \text{ cm.}$$

$$\text{Volume of the cylinder} = \pi r^2 h \text{ cu.units.}$$

$$= \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} \times 7 = \frac{539}{2} = 269.5 \text{ cu.cm}$$

$$\text{Volume} = 269.5 \text{ cu.cm.}$$

Example 2.7 : The radii of two cylinders are in the ratio of 2 : 3, find the ratio of their volumes if their heights are in the ratio of 5 : 3.

Solution :

$$r_1 : r_2 = 2 : 3$$

$$h_1 : h_2 = 5 : 3$$

$$V_1 : V_2 = \pi r_1^2 h_1 : \pi r_2^2 h_2$$

$$= 2 \times 2 \times 5 : 3 \times 3 \times 3$$

$$= 20 : 27$$

$$\text{Ratio of their volumes} = 20 : 27$$

Example 2.8 : The diameter of a roller is 140 cm and 84 cm long. It takes 350 complete revolution to level a play ground, determine the cost of levelling at the rate of Rs. 5 per square metre.

Solution : Diameter = 140 cm

Radius $= \frac{140}{2} = 70$ cm.

Height = 84 cm

Curved surface area $= 2\pi rh$ sq.units.

$= 2 \times \frac{22}{7} \times 70 \times 84 = 36960$ sq.cm.

Area covered in 1 revolution = 36960 sq.cm.

Area covered in 350 revolution $= 36960 \times 350$

$= 12936000$ cm^2

$= \frac{12936000}{100 \times 100} = 1293.6$ m^2

Cost of levelling 1293.6 m^2 $= 1293.6 \times 5 =$ Rs. 6468.

Example 2.9 : A rectangular sheet of metal 44 cm long and 20 cm broad is rolled along its length into a cylinder. Find the curved surface area of the cylinder.

Solution : Length of the sheet = circumference of the base

$44 = 2\pi r$

$44 = 2 \times \frac{22}{7} \times r$

$44 \times \frac{1}{2} \times \frac{7}{22} = r$

$7 = r$

Radius = 7 cm.

Breadth of the metal sheet = height of the cylinder

$h = 20$ cm.

Curved surface area $= 2\pi rh$ sq.units.

$= 2 \times \frac{22}{7} \times 7 \times 20 = 880$ sq.cm.

Curved surface area = 880 sq.cm. **(OR)**

Curved Surface Area = Length × breadth of the rectangular sheet

$= 44 \times 20 = 880$ sq.cm.

Example 2.10 : The radius and height of a cylinder are in the ratio 2 : 7. If the curved surface area of the cylinder is 352 sq.cm, find its radius.

Solution : Let the radius be $2x$ and height be $7x$

$$\text{Curved surface area} = 352 \text{ cm}^2$$

$$2\pi rh = 352$$

$$2 \times \frac{22}{7} \times 2x \times 7x = 352$$

$$88x^2 = 352$$

$$x^2 = \frac{352}{88} = 4$$

$$x = 2$$

$$\text{Radius} = 2 \times 2 = 4 \text{ cm}$$

Example 2.11 : Through a cylindrical tunnel of diameter 21 metres water flows uniformly at the rate of 18 km per hour. How much water will flow through it in 20 minutes.

Solution : Diameter of the cylindrical tunnel = 21 m

$$\text{Radius } (r) = \frac{21}{2} \text{ m}$$

$$\text{Speed of water} = 18 \text{ km/hr}$$

$$= 18000 \text{ m/hr}$$

$$\text{Volume of flowing water in 60 minute} = \pi r^2 h \text{ cu.units.}$$

$$= \frac{22}{7} \times \frac{21}{2} \times \frac{21}{2} \times 18000$$

$$\text{Volume of flowing water in 20 minutes} = \frac{22}{7} \times \frac{21}{2} \times \frac{21}{2} \times 18000 \times \frac{20}{60}$$

$$= 2079000 \text{ cu.m.}$$

Volume of flowing water in 20 minutes = 2079000 cu.m.

Example 2.12 : A cylindrical water tank of diameter 1.4 m and height 2.1 m is being fed by a pipe of diameter 3.5 cm through which water flows at the rate of 2 m/sec. Calculate in minutes the time it takes to fill the tank.

Solution :

Cylindrical tank

Diameter = 1.4 m

Radius (r_1) = 0.7 m

Height (h_1) = 2.1 m

$= \frac{21}{10}$ m

Cylindrical pipe

Diameter = 3.5 cm

$= \frac{35}{10}$ cm

$= \frac{7}{2}$ cm

Radius (r_2) $= \frac{7}{4}$ cm

$= \frac{7}{400}$ m

Speed (h_2) = 2m/sec.

$$\text{Time taken} = \frac{\text{Volume of cylindrical tank}}{\text{Volume of cylindrical pipe}}$$

$$= \frac{\pi r_1^2 h_1}{\pi r_2^2 h_2}$$

$$= \frac{\pi \times \frac{7}{10} \times \frac{7}{10} \times \frac{21}{10}}{\pi \times \frac{7}{400} \times \frac{7}{400} \times 2}$$

$$= \frac{7}{10} \times \frac{7}{10} \times \frac{21}{10} \times \frac{400}{7} \times \frac{400}{7} \times \frac{1}{2}$$

= 1680 seconds

$= \frac{1680}{60}$ min. = 28 minutes.

Time taken = 28 minutes.

Example 2.13 : Water flows through a cylindrical pipe of internal radius 3.5 cm at 5 m per sec. Calculate the volume of water in litres discharged by the pipe in one minute.

Solution : Radius (r) = 3.5 cm $= \frac{35}{10}$ cm

Speed (h) $= 5$m/sec $= 500$ cm/sec.

Volume of flowing water in 1 sec $= \pi r^2 h$ cu.units.

$$= \frac{22}{7} \times \frac{35}{10} \times \frac{35}{10} \times 500$$

Volume of flowing water in 60 sec $= \frac{22}{7} \times \frac{35}{10} \times \frac{35}{10} \times 500 \times 60$

$= 1155000$ cu.cm.

$= \frac{1155000}{1000}$ litres

$= 1155$ litres.

Volume of flowing water in 1 minute $= 1155$ litres.

Example 2.14 : A hollow cylindrical iron pipe is 40 cm long. Its outer and inner diameters are 8 cm and 5 cm respectively. Find the volume of the material, and the weight of the pipe if 1 cc of iron weighs 7 gm.

Solution : Outer radius (R) $= \frac{8}{2} = 4$ cm

Inner radius (r) $= \frac{5}{2} = 2.5$ cm.

Height of the pipe $= 40$ cm

Volume of the material $= \pi h\,(R + r)\,(R - r)$ cu.units.

$$= \frac{22}{7} \times 40\,(4 + 2.5)\,(4 - 2.5)$$

$$= \frac{22}{7} \times 40 \times 6.5 \times 1.5 \text{ cu.cm.}$$

Given weight of the pipe for 1 cu.cm is 7 gm

$\therefore$ weight of the hollow cylindrical iron pipe $= \frac{22}{7} \times 40 \times 6.5 \times 1.5 \times 7$

$= 8580$ gm.

$= 8.58$ Kg.

Weight of the pipe $= 8.58$ Kgs.

EXERCISE 2.1

1. The diameter of a cylinder is 28 cm and its height is 20 cm find

   (a) the curved surface area, (b) total surface area and (c) volume.

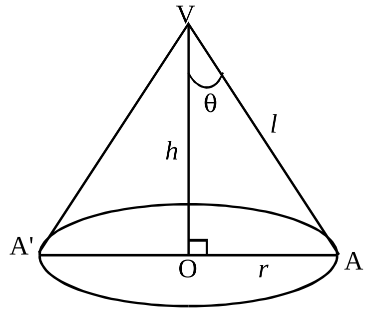
2. The area of the base of a cylinder is 36.75 cm^2 and its height is 12 cm find its volume.
3. How much earth will be excavated from a circular well 17.5 m deep and 2 m base diameter ?
4. 10 cylindrical pillars of building have to be painted. If the diameter of each pillar is 50 m and the height is 4 m, find the cost of painting at the rate of Rs. 7 per square metre.
5. A cylinder of maximum volume is cut out from a solid metal cuboid of length 20 cm and cross section a square side 14 cm. Find the volume of the metal wasted.
6. The ratio between the radius of the base and the height of a cylinder is 2 : 7, find the radius of the cylinder if its volume is 5632 cu.cm.
7. The volume of a cylinder is 98 π cu.cm and its height is 8 cm. Find its lateral surface area.
8. The total surface area of a circular cylinder is 1540 cm^2. If the height is four times the radius of the base, find the height of the cylinder.
9. The area of the curved surface of a cylinder is 4400 sq.cm. and the circumference of its base is 110 cm find the volume of the cylinder.
10. The area of the curved surface of the cylinder is 704 sq.cm. and its height is 8 cm, find the capacity of cylinder in litres.
11. The curved surface of a cylindrical pillar is 264 sq.m and its volume is 924 cu.m. Find the height of the pillar.
12. Through a pipe of diameter 14 cm water flows uniformly at the rate of 3 km per hour. How much water will flow through it in 10 minutes.
13. A lead pencil is in the shape of cylinder. If the pencil is 28 cm long radius 4 mm and its lead is of radius 1 mm, find the volume of the wood used in the pencil.
14. The inner and outer radii of a hollow cylinder are 8 cm and 10 cm respectively. If its outer surface area is 440 sq.cm, find its inner surface area.
15. A hollow cylinder has a total surface area of 1320 sq.cm. If its internal diameter is 8 cm and height is 7 cm, find its external radius.

2.3 RIGHT CIRCULAR CONE

A right circular cone is a solid generated by revolving a line segment which passes through a fixed point and which makes a constant angle with a fixed vertical line.

In the figure, VO is a fixed line, V is the fixed point and VA is the revolving line which makes a constant angle θ with VO.

The fixed point V is called the vertex of the cone, the fixed vertical line is known as its axis and the revolving line VA is called generator of the cone. If only the segment VA revolves about VO, it generates a hollow cone with open base.



If the segments VA and OA of the right triangle OAV or the triangle VA'A revolve about VO, we get a hollow cone with closed circular base with centre O and radius OA. If the plane lamina VOA revolves about VO, we obtain a solid cone.

Height of the cone

The length of the segment VO is called the height of the cone and is generally denoted by '*h*'.

Slant height of the cone

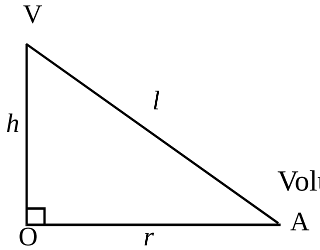
The length of the segment VA is called the slant height of the cone.

In other words, the distance of the vertex from any point on the base of the circle is called the slant height of the cone and is generally denoted by '*l*'.

Radius of the cone

The radius OA of the base circle is called the radius of the cone and is generally denoted by *r*.

It is evident from the figure that



$$VA^2 = VO^2 + OA^2$$

$$l^2 = h^2 + r^2$$

$$l = \sqrt{r^2 + h^2}$$

$$\text{Volume of a cone} = \frac{1}{3} \times \text{base area} \times \text{height}$$

$$\text{Volume of a cone} = \frac{1}{3} \times \pi r^2 h \text{ cu.units.}$$

Surface area of a cone = $\frac{1}{2}$ × perimeter of the base × slant height

$$= \frac{1}{2} \times 2\pi r \times l = \pi rl$$

Curved surface area $= \pi rl$ sq.units.

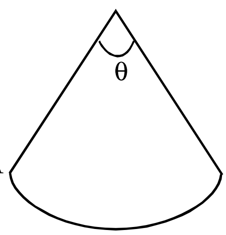
Total surface area = Curved surface area + Base area

$$= \pi rl + \pi r^2 = \pi r(l + r)$$

Total surface area $= \pi r(l + r)$ sq.units.

Hollow cone

If a sheet of paper in the form of a sector of a circle is folded by joining the two bounding radii, we get a figure called hollow cone.



When the sector AOB of a circle is folded by bringing OA and OB together then we obtain a hollow cone. Radius of the sector becomes the slant height of the cone.

Radius of the sector = Slant height

$$R = l$$

Length of the arc AB becomes the circumference of the base of the cone

Length of the arc AB = $\frac{\theta}{360} \times 2\pi R$

where θ is the angle of the sector, R is the radius of the sector

circumference of the base of cone = length of the arc.

$$2\pi r = \frac{\theta}{360} \times 2\pi R$$

Radius of a cone $= r = \frac{\theta}{360}$ × Radius of a sector

$$r = \frac{\theta}{360} \times l$$

Example 2.15 : Find the slant height of the cone whose base has the diameter 10 cm and whose height is 12 cm.

Solution : Diameter of the base of cone = 10 cm

$$\text{Radius } (r) = \frac{10}{2} = 5 \text{ cm}$$

$$\text{Height } (h) = 12 \text{ cm}$$

$$\text{Slant height } l = \sqrt{r^2 + h^2} = \sqrt{(5)^2 + (12)^2}$$

$$= \sqrt{25+144} = \sqrt{169} = 13$$

$$\text{Slant height} = 13 \text{ cm.}$$

Example 2.16 : What is the lateral surface area of a cone whose slant height is 10 cm and base diameter is 14 cm ?

Solution : Radius of the base of cone $(r) = \frac{14}{2} = 7$ cm

$$\text{Slant height } (l) = 10 \text{ cm}$$

$$\text{Lateral surface area} = \pi r l \text{ sq.units}$$

$$= \frac{22}{7} \times 7 \times 10 = 220 \text{ sq.cm.}$$

$$\text{Lateral surface area} = 220 \text{ sq.cm.}$$

Example 2.17 : If the slant height and diameter of a conical tomb are 25 m and 14 m respectively, find the volume of the conical tomb.

Solution : Diameter of the base of cone = 14 m

$$\text{Radius } (r) = \frac{14}{2} = 7 \text{ m}$$

$$\text{Slant height } (l) = 25 \text{ m}$$

$$h = \sqrt{l^2 - r^2}$$

$$= \sqrt{625-49} = \sqrt{576} = 24$$

$$\text{Height} = 24 \text{ m.}$$

$$\text{Volume of the cone} = \frac{1}{3}\pi r^2 h \text{ cu.units.}$$

$$= \frac{1}{3} \times \frac{22}{7} \times 7 \times 7 \times 24 = 1232 \text{ cu.m.}$$

$$\text{Volume of the cone} = 1232 \text{ cu.m.}$$

Example 2.18 : The area of the base right circular cone is 78.5 sq.m. If its height is 12 m find its volume.

Solution : Area of the base of cone = 78.5 sq.m.

height = 12 m

Volume of the cone $= \frac{1}{3} \times$ Base area $\times$ height

$= \frac{1}{3} \times 78.5 \times 12 = 314$ cu.m.

Volume of the cone = 314 cu.m.

Example 2.19 : The volume of a cone is the same as that of cylinder whose height is 9 cm and diameter 40 cm. Find the radius of the base of the cone if its height is 108 cm.

Solution :

Cone	***Cylinder***
Radius r = ?	Radius (r_1) = 20 cm
height h = 108 cm	height (h_1) = 9 cm

Volume of a cone = Volume of a cylinder

$$\frac{1}{3}\pi r^2 h = \pi r_1^2 h_1$$

$$\frac{1}{3} \times r^2 \times 108 = 20 \times 20 \times 9$$

$$r^2 = 20 \times 20 \times 9 \times 3 \times \frac{1}{108}$$

$$r^2 = 100$$

$$r = 10 \text{ cm}$$

Radius of cone = 10 cm

Example 2.20 : A conical tent is required to accommodate 7 persons, each person requires 22 dm^2 of space on the floor and 176 cu.dm of air to breathe find the vertical height.

Solution : Area of space for 1 person = 22 sq.dm.

Area of space for 7 persons = 22 × 7

Base area = 154 sq.dm.

Air space for 1 person = 176 cu.dm.

Air space for 7 persons = 176 × 7

= 1232 cu.dm.

Volume of air in the tent = 1232 cu.dm.

$\frac{1}{3}$ × Base area × height = 1232

$\frac{1}{3}$ × 154 × height = 1232

Height = $1232 \times 3 \times \frac{1}{154}$

= 24

Height = 24 dm.

Example 2.21 : How many metres of cloth 11 m wide will be required to make a conical tent, the radius of whose base is 7 m and whose height is 24 m.

Solution : Radius (r) = 7 m ; height (h) = 24 m

Slant height (l) = $\sqrt{r^2 + h^2}$

= $\sqrt{(7)^2 + (24)^2}$

= $\sqrt{49+576}$ = $\sqrt{625}$ = 25

l = 25 m.

Rectangle	***Cone***
L = ?	r = 7 m
B = 11 m	l = 25 m

Area of rectangular cloth = CSA of cone

LB = $\pi r l$

L × 11 = $\frac{22}{7} \times 7 \times 25$

L = $\frac{22}{7} \times 7 \times 25 \times \frac{1}{11}$

= 50

Length of the cloth = 50 m.

Example 2.22 : The volume of a cone is the same as that of the cylinder whose height is 9 cm and diameter 40 cm. Find the radius of the base of the cone if its height is 108 cm.

Solution :

Cone	***Cylinder***
Radius (r_1) = ?	Diameter = 40 cm
Height (h_1) = 108 cm	Radius r_2 = 20 cm
	Height (h_2) = 9 cm

Volume of cone = Volume of cylinder

$$\frac{1}{3}\pi r_1^2 h_1 = \pi r_2^2 h_2$$

$$\frac{1}{3} \times r_1^2 \times 108 = 20 \times 20 \times 9$$

$$r_1^2 = 20 \times 20 \times 9 \times 3 \times \frac{1}{108}$$

$$r_1^2 = 100$$

$$r_1 = 10$$

Radius of cone = 10 cm.

Example 2.23 : The radius and the height of a right circular cone are in the ratio 5 : 12. If its volume is 314 cu.m. Find the radius and slant height. (Take π = 3.14)

Solution : Given ratio of the radius and height of a cone is 5 : 12

Let the radius be $5x$ m and height be $12x$ m

Volume of cone = 314 cu.m.

$$\frac{1}{3}\pi r^2 h = 314$$

$$\frac{1}{3} \times 3.14 \times 5x \times 5x \times 12x = 314$$

$$314x^3 = 314$$

$$x^3 = 1$$

$$x = 1$$

$\therefore$ Radius of cone $= 5x = 5 \times 1 = 5$ m

and height of cone $= 12x = 12 \times 1 = 12$ m

Slant height (l) $= \sqrt{r^2 + h^2}$

$= \sqrt{(5)^2 + (12)^2}$

$= \sqrt{25+144} = \sqrt{169} = 13$

Slant height = 13 m.

Example 2.24 : The curved surface area of a cone is 550 sq.cm. and the total surface area is 704 sq.cm. Find the radius and height of the cone.

Solution : Curved surface area of cone = 550 sq.cm.

Total surface area = 704 sq.cm.

$$\pi r\,(l + r) = 704$$

$$\pi rl + \pi r^2 = 704$$

$$550 + \pi r^2 = 704$$

$$\pi r^2 = 704 - 550$$

$$\frac{22}{7} \times r^2 = 154$$

$$r^2 = 154 \times \frac{7}{22}$$

$$r^2 = 49$$

$$r = 7 \text{ cm}$$

Curved surface area = 550 sq.cm.

$$\pi rl = 550$$

$$\frac{22}{7} \times 7 \times l = 550$$

$$l = \frac{550}{22} = 25 \text{ cm}$$

Slant height = 25 cm

$$h = \sqrt{l^2 - r^2}$$

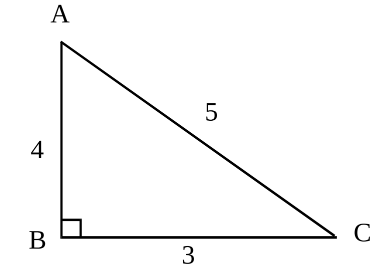
$$= \sqrt{625-49} = \sqrt{576} = 24$$

Height = 24 cm.

Example 2.25 : A right angled triangle whose sides are 3 cm, 4 cm and 5 cm is revolved about the sides containing the right angle in two ways find the difference in volumes of the two cones so formed.

Solution : First case :

Let the triangle be revolved about the side AB



Height of cone = 4 cm

Radius of base = 3 cm

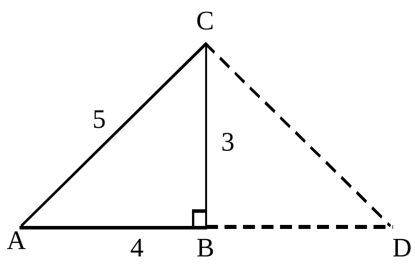
Volume of cone $= \frac{1}{3}\pi r^2 h$ cu.units.

$= \frac{1}{3} \times \pi \times 3 \times 3 \times 4$

$= 12\,\pi$ cu.cm.

Second case :

Let the triangle be revolved about the side BC



Height of cone = 3 cm

Radius of base = 4 cm

Volume of cone $= \frac{1}{3}\pi r^2 h$ cu.units.

$= \frac{1}{3} \times \pi \times 4 \times 4 \times 3$

$= 16\,\pi$ cu.cm.

Difference in two volumes $= 16\,\pi - 12\,\pi$

$= 4\pi$ cu.m.

Example2.26 : A sector of a circle of radius 12 cm has the angle 120°. It is rolled up so that two bounding radii are joined together to form a cone. Find the volume of the cone.

Solution : Angle of the sector (θ) = 120°

Radius of the sector (R) = 12 cm

Circumference of the base of cone = length of the arc

$$2\pi r = \frac{\theta}{360} \times 2\pi R$$

$$r = \frac{\theta}{360} \times R = \frac{120}{360} \times 12 = 4 \text{ cm}$$

Slant height of the cone = Radius of a sector

l = 12 cm

$h = \sqrt{l^2 - r^2}$

$= \sqrt{(12)^2 - (4)^2}$

$= \sqrt{144 - 16} = \sqrt{128} = 11.31$

Height of cone = 11.31 cm

Volume of cone $= \frac{1}{3}\pi r^2 h$ cu.units

$= \frac{1}{3} \times \frac{22}{7} \times 4 \times 4 \times 11.31$

= 189.58 cu.cm.

Volume of cone = 189.58 cu.cm.

Example 2.27 : A sector containing an angle of 90^o is cut from a circle of radius 20 cm and folded into a cone. Find the curved surface area of the cone.

Solution : Angle of the sector (θ) = 90^o

Radius of sector (R) = 20 cm.

Circumference of the base of cone = length of the arc

$2\pi r = \frac{\theta}{360} \times 2\pi R$

$r = \frac{\theta}{360} \times R = \frac{90}{360} \times 20 = 5$

Radius of cone = 5 cm.

Slant height = Radius of a sector

l = 20 cm

Curved surface area = πrl sq.units.

$= \pi \times 5 \times 20 = 100\,\pi$ sq.cm.

Curved surface area = $100\,\pi$ sq.cm.

Example 2.28 : A cone of base radius 5 cm and height 12 cm is opened out into a sector of circle find the central angle of the sector.

Solution : Radius (r) = 5 cm

Height (h) = 12 cm

Slant height (l) $= \sqrt{r^2 + h^2}$

$= \sqrt{(5)^2 + (12)^2}$

$= \sqrt{25+144} = \sqrt{169} = 13$

l = 13 cm

Radius of a sector = slant height

R = 13 cm

Length of the arc = circumference of the base of cone

$\frac{\theta}{360} \times 2\pi R = 2\pi r$

$\frac{\theta}{360} \times 13 = 5$

$\theta = 5 \times \frac{360}{13} = \frac{1800}{13} = 138^o28'$

Central angle of the sector = 138°28'

Example 2.29 : Find the perimeter of the circular sector obtained by opening out a cone of base radius 5 cm and height 12 cm.

Solution : Radius (r) = 5 cm

Height (h) = 12 cm

Slant height (l) $= \sqrt{r^2 + h^2}$

$= \sqrt{(5)^2 + (12)^2}$

$= \sqrt{25+144} = \sqrt{169}$ = 13 cm

Perimeter of a sector $= 2\pi r + 2l$

$= \left(2 \times \frac{22}{7} \times 5\right) + (2 \times 13)$

$= \frac{220}{7} + 26 = 31.43 + 26 = 57.43$

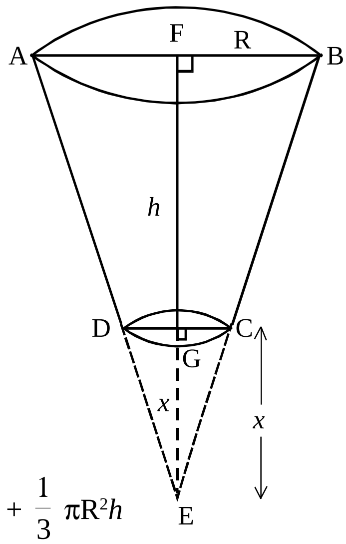
Perimeter of a sector = 57.42 cm.

Volume of Frustum of a cone

A solid got by cutting a right circular cone by a plane parallel to the base is known as **Frustum of a cone.**

Volume of the frustum (V) = $\frac{1}{3}\pi R^2 (x + h) - \frac{1}{3}\pi r^2 x$ cu.units.

Δ BFE ||| Δ CGE



$$\frac{R}{r} = \frac{x+h}{x}$$

$$x = \left(\frac{rh}{R-r}\right)$$

$$\therefore V = \frac{1}{3}\pi R^2 x + \frac{1}{3}\pi R^2 h - \frac{1}{3}\pi r^2 x$$

$$= x \times \frac{1}{3}\pi (R^2 - r^2) + \frac{1}{3}\pi R^2 h$$

$$= \frac{rh}{(R-r)} \times \frac{1}{3}\pi (R + r)(R - r) + \frac{1}{3}\pi R^2 h$$

$$= \frac{1}{3}\pi rh (R + r) + \frac{1}{3}\pi R^2 h$$

$$= \frac{1}{3}\pi h (R^2 + Rr + r^2) \text{ cubic units.}$$

Example 2.30 : Find the capacity of a bucket having the radius of the top as 36 cm and that of the bottom as 12 cm. Its depth is 35 cm.

Solution : R = 36 cm, r = 12 cm, h = 35 cm

$$V = \frac{1}{3}\pi h (R^2 + Rr + r^2)$$

$$V = \frac{1}{3} \times \frac{22}{7} \times 35\, [36^2 + (36)(12) + 12^2] \text{ cu.cm.}$$

$$= \frac{1}{3} \times \frac{22}{7} \times 35 \times 1872 = 68640 \text{ cu.cm.}$$

$\therefore$ Capacity of the bucket = 68640 cm^3.

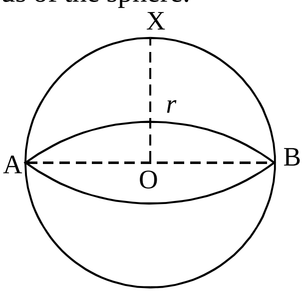
EXERCISE - 2.2

1. The slant height of a circular cone is 29 cm and height is 20 cm. What is the radius of the base of cone ?
2. What is the lateral surface area of a cone when slant height is 10 cm and height is 8 cm ?
3. Find the volume of the cone given that its slant height is 17 cm and radius is 8 cm.
4. The circumference of the base of a tent is 17.6 m. The slant height is 3.5 m find the area of the canvas used for the tent.
5. The volume of a cone is 1232 cu.cm. Determine the area of the base if its height is 24 cm.
6. The base area of a cone is 9π sq.cm and its slant height is 5 cm. What is the height of the cone ?
7. A conical tent of 56 m base diameter requires 3080 sq.m of canvas for the curved surface. Find its height.
8. Two cones have their height in the ratio 5 : 3 and the radii of their bases in the ratio 2 : 1, find the ratio of their volumes.
9. A conical tent is to accommodate 6 persons, each person have 20 sq.m. of space on the ground and 120 cu.m of air to breathe. Find the height of the cone.
10. A right circular cone of height 40 cm and base radius 15 cm is casted into smaller cones of equal sizes with base radius 5 cm and height 4 cm. Find how many cones are made.
11. The two sides making a right angle of a right angled triangle are 5 cm and 12 cm. It is revolved about the bigger side of these. Find the volume of the cone thus formed.
12. A sheet of metal in the shape of quadrant of a circle of radius 28 cm is bent into an open cone. Find the curved surface area of the cone.
13. A semicircular plate of tin has a diameter of 40 cm. It is made into an open conical vessel by bringing the radii together and sholdering, find the capacity of the vessel.
14. Find the length of arc of the sector formed by opening out a cone of base radius 8 cm. What is the central angle, if the height of the cone is 6 cm ?
15. The curved surface area of the cone is 814 sq.cm. and the total surface area of the cone is 1584 sq.cm. Find its volume ?
16. The diameter of the bottom of a bucket is 14.4 cm and that of the top is 43.2 cm. Its depth is 28 cm. Find the capacity of the bucket. ($\pi = \frac{22}{7}$)

2.4 SPHERE

A sphere is a solid described by the revolution of a semi circle about its diameter, which remains fixed.

In the figure, AOB is the diameter. The sphere shown is generated by one complete revolution of the semicircle AXB about the diameter AOB. The centre and radius of the semicircle are the centre and radius of the sphere.



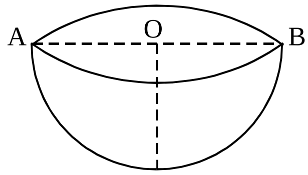
Let r be the radius of the sphere

Volume of the sphere = $\frac{4}{3}\pi r^3$ cu.units.

Surface of the sphere = $4\pi r^2$ sq.units.

HEMISPHERE

When the sphere is cut by a plane through its centre into two equal parts each part is called the hemisphere.



Let r be the radius of the hemisphere.

1. Volume of the hemisphere = $\frac{2}{3}\pi r^3$ cu.units

2. Curved surface area = $2\pi r^2$ sq.units

3. Total surface area of a solid hemisphere

= curved surface area + area of circular base

$= 2\pi r^2 + \pi r^2 = 3\pi r^2$

Total surface area of a solid hemisphere = $3\pi r^2$ sq.units.

Example 2.31 : If the diameter of a sphere is 7 cm, find (i) its surface area and (ii) its volume.

Solution : Diameter = 7 cm

Radius (r) = $\frac{7}{2}$ cm

(i) Surface area = $4\pi r^2$ sq.units.

$= 4 \times \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2}$ = 154 sq.cm.

Surface area $= 154$ sq.cm.

(ii) Volume $= \frac{4}{3}\pi r^3$ cu.units

$$= \frac{4}{3} \times \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} \times \frac{7}{2} = 179.67 \text{ cu.cm}$$

Volume $= 179.67$ cu.cm.

Example 2.32 : The surface area of a sphere is 2464 sq.cm. Find its radius.

Solution : Surface area $= 2464$ sq.cm.

$$4\pi r^2 = 2464$$

$$4 \times \frac{22}{7} \times r^2 = 2464$$

$$r^2 = 2464 \times \frac{1}{4} \times \frac{7}{22}$$

$$r^2 = 196$$

$$r = 14 \text{ cm}$$

Example 2.33 : The surface areas of the two spheres are in the ratio 25 : 36 find the ratio of their radii.

Solution : Let their radii be r_1 and r_2.

Let A_1 and A_2 be the surface areas of two spheres.

$$A_1 : A_2 = 25 : 36$$

$$4\pi r_1^2 : 4\pi r_2^2 = 25 : 36$$

$$r_1^2 : r_2^2 = 25 : 36$$

$$r_1 : r_2 = 5 : 6$$

Ratio of their radii $= 5 : 6$

Example 2.34 : The volume of the sphere is numerically equal to its surface area. Find its diameter.

Solution :Volume of surface = Surface area of sphere

$$\frac{4}{3}\pi r^3 = 4\pi r^2 \qquad \Rightarrow \frac{r}{3} = 1$$

$$\therefore r = 3 \text{ cm.}$$

Diameter of sphere $= 2 \times 3 = 6$ cm.

Example 2.35 : A solid metal cylinder of radius 14 cm and height 21 cm is melted down and recast into sphere of diameter 7 cm. Calculate the number of spheres that can be made.

Solution :

Cylinder	***Sphere***
Radius = 14 cm	Diameter = 7 cm
Height = 21 cm	Radius $= \frac{7}{2}$ cm

$$\text{No. of spheres} = \frac{\text{Volume of cylinder}}{\text{Volume of sphere}}$$

$$= \frac{\pi r^2 h}{\frac{4}{3}\pi r_1^3}$$

$$= \frac{14 \times 14 \times 21}{\frac{4}{3} \times \frac{7}{2} \times \frac{7}{2} \times \frac{7}{2}}$$

$$= 14 \times 14 \times 21 \times \frac{3}{4} \times \frac{2}{7} \times \frac{2}{7} \times \frac{2}{7} = 72$$

Number of spheres = 72.

Example 2.36 : An iron sphere of diameter 12 cm is dropped into a cylindrical can of diameter 24 cm containing water. Find the rise in the level of water when the sphere is completely immersed.

Solution : Diameter of a sphere = 12 cm

Radius of sphere (r) = 6 cm

Diameter of cylinder = 24 cm

Radius (R) = 12 cm.

Increase in volume of water = Volume of sphere

$$\pi R^2 h = \frac{4}{3}\pi r^3$$

$$12 \times 12 \times h = \frac{4}{3} \times 6 \times 6 \times 6$$

$$h = \frac{4 \times 6 \times 6 \times 6}{3 \times 12 \times 12} = 2 \text{ cm.}$$

Example 2.37 : A hemisphere is of radius 9 cm find the volume and curved surface area.

Solution : Radius of hemisphere (r) = 9 cm.

Volume of hemisphere $= \frac{2}{3} \times \pi r^3$ cu.units

$= \frac{2}{3} \times 9 \times 9 \times 9 \times \pi$

$= 486\ \pi$ cu.cm.

Volume of hemisphere $= 486\ \pi$ cu.cm.

Curved surface area $= 2\pi r^2$ sq.units.

$= 2\pi \times 9 \times 9 = 162\ \pi$

Curved surface area $= 162\ \pi$ sq.cm.

Example 2.38 : The circumference of the edge of a hemispherical bowl is 132 cm. Find its capacity.

Solution : Circumferences the edge of hemisphere = 132 cm

$2\pi r = 132$

$2 \times \frac{22}{7} \times r = 132$

$r = 132 \times \frac{1}{2} \times \frac{7}{22} = 21$ cm

Radius = 21 cm

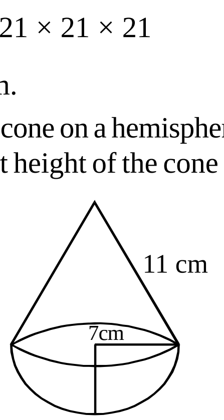
Volume of hemisphere $= \frac{2}{3}\ \pi r^3$ cu.units.

$= \frac{2}{3} \times \frac{22}{7} \times 21 \times 21 \times 21$

= 19404 cu.cm.

Example 2.39 : A plastic doll is made by surmounting a cone on a hemisphere of equal radius. The radius of the hemisphere is 7 cm and slant height of the cone is 11 cm. Find the surface area of the doll.

Solution :



Cone	***Hemisphere***
Radius r = 7 cm	r = 7 cm
Slant height l = 11 cm	

Surface area of hemisphere $= 2\pi r^2$ sq.units

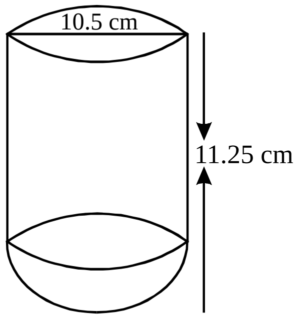
$$= 2 \times \frac{22}{7} \times 7 \times 7 = 308 \text{ sq.cm.}$$

Curved surface area of cone $= \pi rl$ sq.units.

$$= \frac{22}{7} \times 7 \times 11 = 242 \text{ sq.cm.}$$

Surface area of the doll $= 308 + 242 = 550$ sq.cm.

Example 2.40 : A silver cup has the shape of a hemisphere surmounted by a cylinder. The diameter of the sphere is 10.5 cm and total height of the cup is 11.25 cm find its capacity.



Solution : Radius $= \frac{10.5}{2} = 5.25$ cm

Total height $= 11.25$ cm

Height of cylinder $= 11.25 - 5.25$

$= 6$ cm

Radius $= 5.25 \text{ cm} = 5\frac{1}{4} = \frac{21}{4}$ cm.

Total capacity = Capacity of hemisphere + Capacity of cylinder

$$= \frac{2}{3}\pi r^3 + \pi r^2 h$$

$$= \pi r^2 \left(\frac{2}{3} r + h\right)$$

$$= \frac{22}{7} \times \frac{21}{4} \times \frac{21}{4} \left[\left(\frac{2}{3} \times \frac{21}{4}\right) + 6\right]$$

$$= \frac{22}{7} \times \frac{21}{4} \times \frac{21}{4} \left(\frac{7}{2} + 6\right)$$

$$= \frac{22}{7} \times \frac{21}{4} \times \frac{21}{4} \times \frac{19}{2} = \frac{13167}{16} = 822\frac{15}{16} \text{ cu.cm.}$$

$\therefore$ Capacity of the cup $= 822\frac{15}{16}$ cu.cm.

Example 2.41 : A hollow spherical shell is made of a metal of Weight 4.2 gm per cubic cm. If its internal and external radii are 10 cm and 11 cm find its weight.

Solution : Internal radius (r) = 10 cm

External radius (R) = 11 cm

$$\text{Volume of hollow sphere} = \frac{4}{3}\pi(R^3 - r^3) \text{ cu.units.}$$

$$= \frac{4}{3} \times \frac{22}{7} \times (11^3 - 10^3)$$

$$= \frac{4}{3} \times \frac{22}{7} (1331 - 1000)$$

$$= \frac{4}{3} \times \frac{22}{7} \times 331 \text{ cu.cm.}$$

$$\text{Weight of the metal} = \frac{4}{3} \times \frac{22}{7} \times 331 \times 4.2 = 5825.6 \text{ gm.}$$

Example 2.42 : A hollow sphere of internal and external diameter 4 cm and 8 cm respectively is melted and cast into a cone diameter 8 cm. Find the height of the cone.

Solution :

Cone	***Hollow sphere***
Radius (r_1) = 4 cm	Internal radius (r) = 2 cm
height = h cm	External radius (R) = 4 cm

Volume of cone = Volume of a hollow sphere

$$\frac{1}{3}\pi r_1^2 h = \frac{4}{3} \times \pi(R^3 - r^3)$$

$$\frac{1}{3} \times 4 \times 4 \times h = \frac{4}{3}(4^3 - 2^3)$$

$$\frac{16h}{3} = \frac{4}{3}(64 - 8)$$

$$\frac{16h}{3} = \frac{4}{3} \times 56$$

$$h = \frac{4}{3} \times 56 \times \frac{3}{16} = 14 \text{ cm.}$$

Height of the cone = 14 cm.

Example 2.43 : A hollow spherical shell has an inner radius of 8 cm. If the volume of the material is $\frac{1952\pi}{3}$ c.c, find the thickness of the shell.

Solution : Inner radius (r) = 8 cm

$$\text{Volume of the material} = \frac{1952\pi}{3} \text{ cu.cm.}$$

$$\frac{4}{3}\pi(R^3 - r^3) = \frac{1952\pi}{3}$$

$$\frac{4}{3}\pi(R^3 - 8^3) = \frac{1952\pi}{3}$$

$$R^3 - 512 = \frac{1952}{3} \times \frac{3}{4}$$

$$R^3 - 512 = 488$$

$$R^3 = 488 + 512 = 1000$$

$$R = 10 \text{ cm.}$$

External radius = 10 cm

Thickness = $R - r = 10 - 8 = 2$ cm

Thickness = 2 cm.

EXERCISE 2.3

1. Find the volume and surface area of a sphere of radius (a) 10.5 cm, (b) 28 cm, (c) 7 cm, (d) 14 cm.
2. The surface area of a sphere is 1386 sq.cm. Find its volume.
3. The ratio of the diameters of two spheres is 4 : 5 find the ratio of their volumes.
4. How many litres of water will a hemispherical tank hold, whose diameter is 4.2 m?
5. 8 metallic sphere each of radius 2 cm are melted and cast into a single sphere. Calculate the radius of the new sphere.
6. A vessel is in the form of an inverted cone. Its height is 8 cm and the radius of its top which is open is 5 cm. It is filled with water up to the rim. When lead shots, each of

which is a sphere of diameter 1 cm are dropped into vessel, one fourth of the water flows out. Find the number of lead shots dropped into the vessel.

7. The diameter of a copper sphere is 6 cm. The sphere is melted and is drawn into a long wire of uniform diameter. If the length of the wire is 36 cm, find its radius.

8. A cylindrical bowl with base diameter 7 m contains water. A solid sphere is dropped into it. The water level increases by $4\frac{2}{3}$ m. Find the radius of the sphere.

9. An iron cone of diameter 8 cm and height 12 cm is melted and recast into lead shots of radius 2 mm. How many lead shots are obtained ?

10. A hemispherical bowl of internal diameter 36 cm contains a liquid. This liquid is to be filled into cylindrical shaped bottles of radius 3 cm and height 6 cm. How many bottles are required to empty the bowl ?

11. The height and radius of a cone are equal. The volume of the cone is 35 cu.cm. Find the volume of the sphere whose radius is equal to the height of the cone.

12. A cup has the shape of a hemisphere surmounted by a cylinder. The diameter of the hemisphere is 6 cm. The total height of the cup is 13 cm, find its volume.

13. A cylindrical boiler 2m high and 3.5 m in diameter, has a hemispherical lid, find the volume of the interior of the boiler, including the part covered by the lid.

14. A solid is in the form of a cylinder with hemispherical ends. The total height of the solid is 19 cm and the diameter of the cylinder is 7 cm. Find the total surface area of the solid.

15. A hemispherical shell is made of metal which weighs 3 gm per cubic centimeter of metal. Find the weight of the hemisphere if its internal diameter is 10 cm and has thickness 1 cm.

16. The radii of the internal and external surfaces of a hollow spherical shell are 3 cm and 5 cm respectively. If it is melted and recast into a solid cylinder of height $2\frac{2}{3}$ cm. Find the diameter of the cylinder.

17. A hemispherical bowl has volume of material $\frac{122\pi}{3}$ cc. Its external diameter is 10 cm, find its thickness.

3. SET LANGUAGE

3.1. SETS

3.1.1 Introduction :

Sets underlie other Mathematical topics, such as Logic and Abstract Algebra. In fact, the book 'Elements de Mathematique' written by a group of French Mathematicians under the Pseudonym Nicolas Bourbaki states, ''Now-a-days, it is possible, logically speaking, to device the whole known Mathematics from a single source - the theory of sets''.

Here our concern is **Set Language.**

We have learnt what a set is. Let us have a rapid revision.

A set is a collection of well defined, distinct objects.

In the set A = $\{-5, -1, 0, 2, 3\}$, -5 is an element whereas 4 is not an element.

We denote these as $-5 \in A$, $4 \notin A$ respectively.

We have seen that the sets can be represented in three forms, namely, **Roster method, Descriptive form** and **Rule method.**

I. Roster method : V = $\{a, e, i, o, u\}$

Descriptive form : V = {vowels of English alphabet}

Rule method : V = $\{x/x$ is a vowel of English alphabet$\}$

II. Roster Method : A = $\{1, 2, 3, 4, 5, 6\}$

Descriptive form : A = {positive integers less than 7}

Rule method : A = $\{x/x < 7, x \in N\}$

The Empty set (The null set or the void set)

The set which contains no element is called the empty set. It is denoted as ϕ or { }.

Example : $\{x/x + 6 = 2, x \in N\}$

Subset : Set P is a subset of a set Q, symbolised by $P \subseteq Q$, if and only if all the elements of P are also the element of Q.

We know that every set is a subset of itself. The empty set is a subset of every set.

Proper and improper subsets

Set 'G' is a **proper** subset of H, symbolised by $G \subset H$, if and only if all the elements of set G are elements of set H and set G $\neq$ set H.

That is the set H must contain at least one element not in the set G.

Set S is an **Improper subset** of T, symbolised by $S \subseteq T$, if and only if all the elements of set S are the elements of set T and set S = set T.

Universal Set : A universal set is a set that contains all the elements for any specific discussion. The notation for the universal set is $\bigcup$ (ξ or Ω).

Equal sets : Set C is equal to set D, if and only if set C and set D contain exactly the same elements.

We write this as $C = D$

Equivalent sets : Two sets are said to be equivalent sets if they contain the same number of elements.

If those sets are P and Q, then $P \leftrightarrow Q$.

Disjoint sets : When sets A and B are disjoint, they have no elements in common.

Overlapping sets : When sets C and D have elements, (element) in common, they are known as overlapping sets.

3.1.2 OPERATION ON SETS

We have learnt the following :

Union of sets : $A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ or } x \in A \text{ and } B\}$

Intersection of sets : $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

Complement of a set : The complement of a set (A^1 or A^C or $\overline{A}$) is the set of all elements in the universal set that are not in the set A.

$A' = \{x \mid x \in \bigcup \text{ and } x \notin A\}$

Set Difference : The difference of two sets A and B is the set of elements which belong to A but do not belong to B. It is denoted as A – B.

$A - B = \{x \mid x \in A \text{ and } x \notin B\}$

I. Properties of union

1. Set union is **Commutative**

   $A \cup B = B \cup A$

2. Set union is **Associative.**

   $A \cup (B \cup C) = (A \cup B) \cup C$

II. Properties of intersection

1. Set intersection is **Commutative.**

   $A \cap B = B \cap A$

2. Set intersection is **Associative**

   $A \cap (B \cap C) = (A \cap B) \cap C$

III. Properties of set difference

1. Set difference is not Commutative

   $A - B \neq B - A$

2. Set difference is not Associative

   $A - (B - C) \neq (A - B) - C$

IV. Distributive property

1. Union is distributed over intersection.

   $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

2. Intersection is distributed over union.

   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Let us verify these distributive properties.

Example 3.1 : Given $A = \{x \,/\, -4 \leq x < 3, x \in Z\}$

$B = \{-3, -1, 4, 5, 6\}$ and $C = \{-4, -3, 5, 7, 8\}$

Verify (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution : Given $A = \{-4, -3, -2, -1, 0, 1, 2\}$, $B = \{-3, -1, 4, 5, 6\}$ and $C = \{-4, -3, 5, 7, 8\}$

(i)$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

LHS : $A \cup (B \cap C)$

$(B \cap C) = \{-3, 5\}$

$\therefore$ $A \cup (B \cap C) = \{-4, -3, -2, -1, 0, 1, 2, 5\}$ ... (1)

RHS : $(A \cup B) \cap (A \cup C)$

$A \cup B = \{-4, -3, -2, -1, 0, 1, 2, 4, 5, 6\}$

$A \cup C = \{-4, -3, -2, -1, 0, 1, 2, 5, 7, 8\}$

$\therefore$ $(A \cup B) \cap (A \cup C) = \{-4, -3, -2, -1, 0, 1, 2, 5\}$ ... (2)

From (1) and (2) it is clear that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Hence verified.

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

LHS : $A \cap (B \cup C)$

$B \cup C = \{-3, -1, 4, 5, 6, -4, 7, 8\}$

$\therefore A \cap (B \cup C) = \{-3, -1, -4\}$ ... (1)

RHS : $(A \cap B) \cup (A \cap C)$

$A \cap B = \{-3, -1\}$

$A \cap C = \{-4, -3\}$

$(A \cap B) \cup (A \cap C) = \{-3, -1, -4\}$ ... (2)

From (1) and (2), it is clear $A \cap (B \cup C) = (A \cap B) (A \cap C)$. Hence verified.

3.2 De Morgan's laws

Augustus De Morgan (1806 - 1871) the son of a member of the East India company was born in Madurai, India and educated at Trinity College, Cambridge. De Morgan expressed mathematically the laws which were expressed verbally by William of Ockham in the fourteenth century.

Regarding complementation

(i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

Regarding set difference

(iii) $A - (B \cup C) = (A - B) \cap (A - C)$

(iv) $A - (B \cap C) = (A - B) \cup (A - C)$

Let us verify each of the laws .

Example 3.2: Given $\cup$ = {1, 2, 3, ... 10}, A = {3, 4, 6, 10}, B = {1, 2, 4, 5, 6, 8} Verify De Morgan's laws of complementation.

(i) (A ∪ B)' = A' ∩ B'

LHS : (A ∪ B)'

$$A \cup B = \{1, 2, 3, 4, 5, 6, 8, 10\}$$

$$(A \cup B)' = \{7, 9\} \quad \ldots (1)$$

RHS : A' ∩ B'

$$A' = \{1, 2, 5, 7, 8, 9\}$$

$$B' = \{3, 7, 9, 10\}$$

$$A' \cap B' = \{7, 9\} \quad \ldots (2)$$

From (1) and (2), it is clear (A ∪ B)' = A' ∩ B'

Hence verified.

(ii) (A ∩ B)' = A' ∪ B'

$$\text{LHS} = (A \cap B)'$$

$$A \cap B = \{4, 6\}$$

$$(A \cap B)' = \{1, 2, 3, 5, 7, 8, 9, 10\} \quad \ldots (1)$$

$$\text{Now} \quad A' = \{1, 2, 5, 7, 8, 9\}$$

$$B' = \{3, 7,9, 10\}$$

$$\therefore \text{RHS} = A' \cup B' = \{1, 2, 3, 5, 7, 8, 9, 10\} \quad \ldots (2)$$

From (1) and (2) it is clear that (A ∩ B)' = A' ∪ B'

Hence verified.

Example 3.3 : Given

$$A = \{-8, -7, -5, 1, 2, 4\}$$

$$B = \{-7, 1, 3, 4, 5, 6\}$$

$$C = \{-8, -5, 2, 4, 6, 7\}$$

$$\text{Verify } A - (B \cup C) = (A - B) \cap (A - C)$$

LHS : A – (B ∪ C)

$$B \cup C = \{-7, 1, 3, 4, 5, 6, -8, -5, 2, 7\}$$

$$\therefore A - (B \cup C) = \{\ \} \quad \ldots (1)$$

RHS : (A – B) ∩ (A – C)

Now, A – B = {–8, –5, 2}

A – C = { 1, –7}

(A – B) ∩ (A – C) = { } ... (2)

From (1) and (2), it is clear that A – (B ∪ C) = (A – B) ∩ (A – C)

Hence verified.

Example 3.4 : Given A = {–9, –7, –6, –3, 0, 2}, B = {–7, –3, 0, 4, 5, 6}

C = {–9, –6, 2, –7, 8} verify A – (B ∩ C) = (A – B) ∪ (A – C)

LHS : A – (B ∩ C)

B ∩ C = {–7}

∴ A – (B ∩ C) = {–9, –6, –3, 0, 2} ... (1)

Now, A – B = {–9, –6, 2}

A – C = {–3, 0}

RHS : (A – B) ∪ (A – C) = {–9, –6, 2, –3, 0} ... (2)

From (1) and (2), it is a clear A – (B ∩ C) = (A – B) ∪ (A – C)

Hence verified.

Note : When a universal set is given, only the elements in the universal set may be considered when working the problem. If, for example, the universal set for a particular problem is defined as U = {1, 2, 3, ..., 10}, then only the natural numbers 1 through 10 may be used in that problem.

EXERCISE 3.1

1. Given ⋃ = {1, 2, 3, ..., 15}, A = {2, 3, 7, 8, 11} B = {1, 3, 8, 11, 13, 15}, verify De Morgan's laws regarding complementation.
2. Given ⋃ = {*x*/*x* is a positive divisor of 60}

   A = {*x* / *x* is a positive divisor of 30}

   B = {*x* / *x* is a multiple of 3}

   verify (i) (A ∩ B)' = A' ∪ B' and (ii) A' ∩ B' = (A ∪ B)'
3. Given : A = {2, 3, 5, 6, 8}, B = {2, 4, 7, 9, 10} and C = {3, 4, 5, 7, 10, 11} verify

   (i) A – (B ∪ C) = (A – B) ∩ (A – C)

   (ii) A – (B ∩ C) = (A – B) ∪ (A – C)

4. Given : A = {x / –4 < x ≤ 4, x ∈ Z}

B = {5, 4, 3, 1, 0, –1} and C = {–2, –1, 0, 3, 4}

Verify De Morgan's laws in the case of set difference.

5. Given A = {–5, –2, –1, 0, 1, 3, 4}, B = {–2, 0, 3, 5, 6, 7}

and C = {–5, –1,0, 4, 7, 8}

Verify (i) A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C)

(ii) A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)

3.3 VENN DIAGRAM

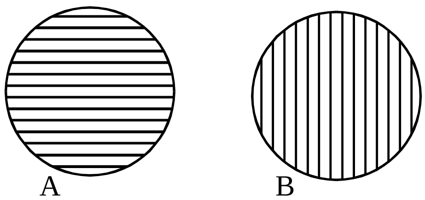
A useful way to represent relationships between sets is to let the universal set be represented by a rectangle, with the proper subsets in the universe represented by circular or oval shaped region.

These pictures are called **Venn diagrams**, after John Venn (1834 - 1923). The Swiss Mathematician Leonhard Euler (1707 - 1783) also used circles to illustrate principles of logic, so sometimes these diagrams are called Euler circles. However, Venn was the first person to use them in a general way.

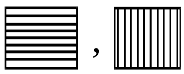
We have learnt the following :

I. Union of 2 sets

Case (i)

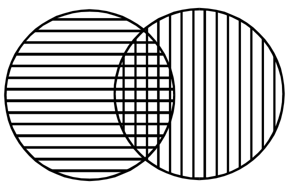


Disjoint sets - Union

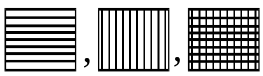


represents A ∪ B

Case (ii)

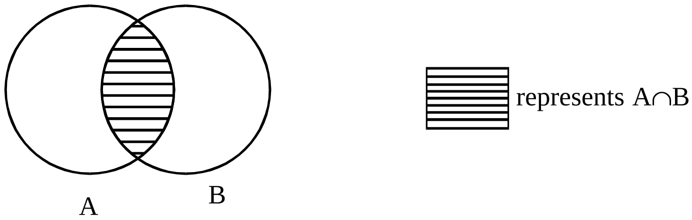


Overlapping sets - union

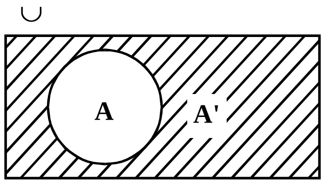


represents A ∪ B

II. Intersection of 2 sets



III. Complement of a set

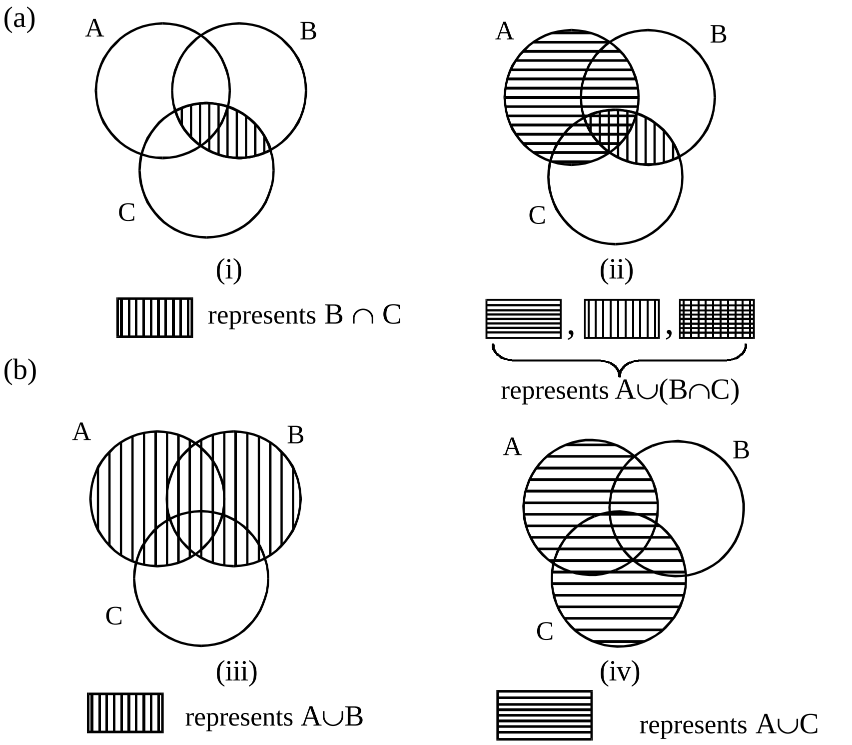


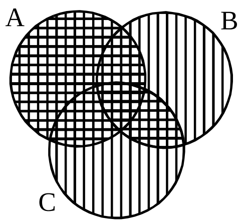
The complement of a set A denoted by A' is the set of all elements in ⋃ that are not in the set A.

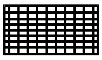
$A' = \{x \,/\, x \in \bigcup ,\ x \notin A\}$

IV. Combined operations with sets

Example 3.5 : Using Venn diagram, verify A∪(B∩C) = (A∪B)∩(A∪C)





 represents (A∪B)∩(A∪C)

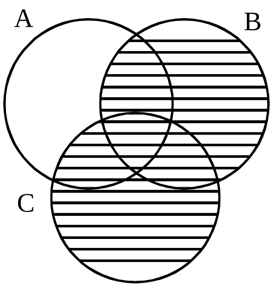
(v)

From figures (ii) and (v) it is clear that,

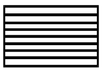
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

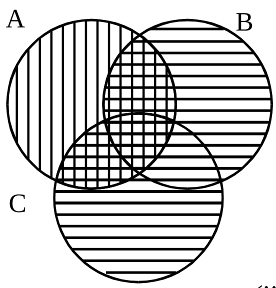
Hence verified.

Example 3.6 : Using Venn diagram, verify $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

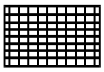


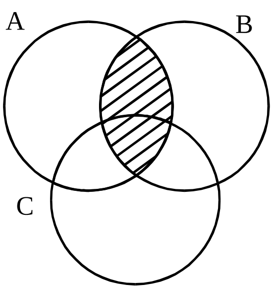
(i)

 represents B∪C

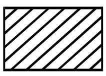


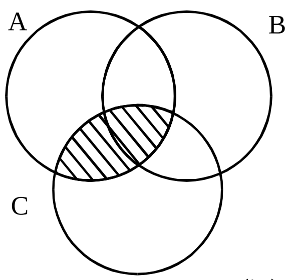
(ii)

 represents A∩(B∪C)

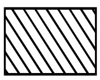


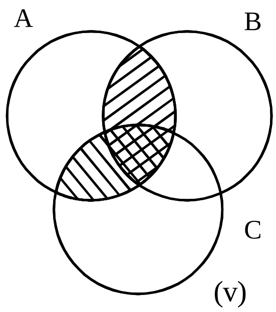
(iii)

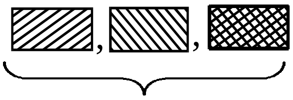
 represents A ∩ B



(iv)

 represents A ∩ C





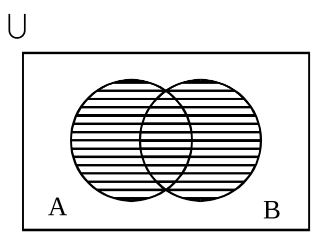
represents (A ∩ B) ∪ (A ∩ C)

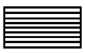
(v) From (ii) and (v), it is clear that

A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C) Hence proved.

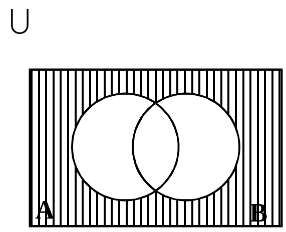
Example 3.7 : Verify De Morgan’s laws regarding complementation using Venn diagram.

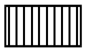
(1) (A ∪ B)' = A' ∩ B'



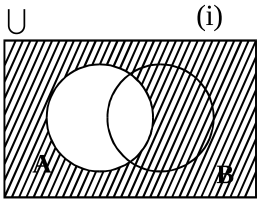
 represents A∪B

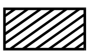
(i)



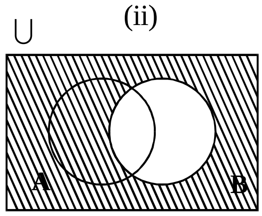
 represents (A∪B)'

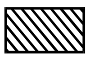
(ii)



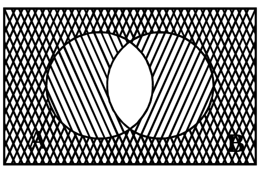
 represents A'

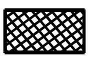
(iii)



 represents B'

(iv)



 represents A' ∩ B'

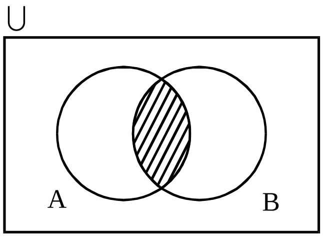
(v)

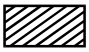
From (ii) and (v) it is very clear.

(A ∪ B)' = A' ∩ B'

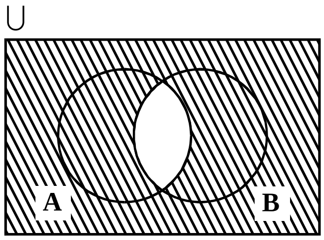
Hence verified.

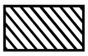
2. $(A \cap B)' = A' \cup B'$.



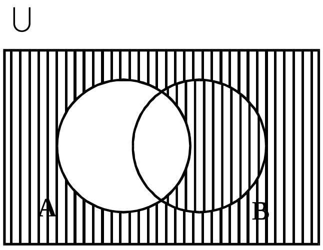
 represents $A \cap B$

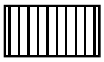
(i)



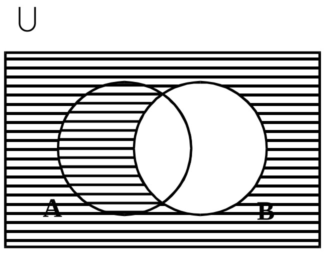
 represents $(A \cap B)'$

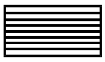
(ii)



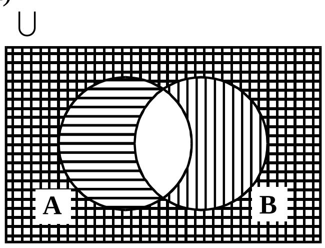
 represents A'

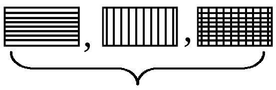
(iii)



 represents B'

(iv)





represents $A' \cup B'$

(v)

From (ii) and (v) it is clear that $(A \cap B)' = A' \cup B'$

EXERCISE 3.2

1. Using Venn diagram verify each of the following :

(1) $(A \cup B)' = A' \cap B'$

(2) $(A \cap B)' = A' \cup B'$

(3) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(4) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(5) $A - (B \cup C) = (A - B) \cap (A - C)$

(6) $A - (B \cap C) = (A - B) \cup (A - C)$

3.4 MEMBERSHIP TABLE METHOD

Before entering into this method, let us consider a situation.

Suppose, we want to meet A, B, C in a school. The time is 4 PM. The school bell has gone. What are the possible situations ?

If we are lucky, we may meet all the three. One may be absent or two may be absent or all the three may be absent. The following table illustrates all these possibilities.

A	B	C
$\in$	$\in$	$\in$
$\in$	$\in$	$\notin$
$\in$	$\notin$	$\in$
$\notin$	$\in$	$\in$
$\in$	$\notin$	$\notin$
$\notin$	$\in$	$\notin$
$\notin$	$\notin$	$\in$
$\notin$	$\notin$	$\notin$

$\in$ – present, $\notin$ – absent

We know $A \cup B = \{x / x \in A \text{ or } x \in B \text{ or } x \in A \text{ and } B\}$

$A \cap B = \{x/x \in A \text{ and } B\}$

$A - B = \{x/x \in A, \text{ but } x \notin B\}$

$A' = \{x / x \in \bigcup, x \notin A\}$

Example 3.8 : Using membership table prove the following :
(i) $(A \cup B)' = A' \cap B'$, (ii) $A - B = A \cap B'$, (iii) $A-(B \cup C) = (A - B) \cap (A - C)$ and (iv) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

1. (A ∪ B)' = A' ∩ B'

∪	A	B	A ∪ B	(A ∪ B)'	A'	B'	A' ∩ B'
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
∈	∈	∈	∈	∉	∉	∉	∉
∈	∈	∉	∈	∉	∉	∈	∉
∈	∉	∈	∈	∉	∈	∉	∉
∈	∉	∉	∉	∈	∈	∈	∈

From (5) and (8), we see that (A ∪ B)' = A' ∩ B'

Hence, we have proved that (A ∪ B)' = A' ∩ B'

2. A – B = A ∩ B'

∪	A	B	A – B	B'	A ∩ B'
(1)	(2)	(3)	(4)	(5)	(6)
∈	∈	∈	∉	∉	∉
∈	∈	∉	∈	∈	∈
∈	∉	∈	∉	∉	∉
∈	∉	∉	∉	∈	∉

From (4) and (6) we see that A – B = A ∩ B'

Hence, we have proved that A – B = A ∩ B'

3. A – (B ∪ C) = (A – B) ∩ (A – C) - prove.

A	B	C	B ∪ C	A – (B ∪ C)	A – B	A – C	(A – B) ∩ (A – C)
1	2	3	4	5	6	7	8
∈	∈	∈	∈	∉	∉	∉	∉
∈	∈	∉	∈	∉	∉	∈	∉
∈	∉	∈	∈	∉	∈	∉	∉
∉	∈	∈	∈	∉	∉	∉	∉
∈	∉	∉	∉	∈	∈	∈	∈
∉	∈	∉	∈	∉	∉	∉	∉
∉	∉	∈	∈	∉	∉	∉	∉
∉	∉	∉	∉	∉	∉	∉	∉

From (5) and (8), we get $A - (B \cup C) = (A - B) \cap (A - C)$

Hence, we have proved $A - (B \cup C) = (A - B) \cap (A - C)$

4. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	2	3	4	5	6	7	8
$\in$	$\in$	$\in$	$\in$	$\in$	$\in$	$\in$	$\in$
$\in$	$\in$	$\notin$	$\in$	$\in$	$\in$	$\notin$	$\in$
$\in$	$\notin$	$\in$	$\in$	$\in$	$\notin$	$\in$	$\in$
$\notin$	$\in$	$\in$	$\in$	$\notin$	$\notin$	$\notin$	$\notin$
$\in$	$\notin$	$\notin$	$\notin$	$\notin$	$\notin$	$\notin$	$\notin$
$\notin$	$\in$	$\notin$	$\in$	$\notin$	$\notin$	$\notin$	$\notin$
$\notin$	$\notin$	$\in$	$\in$	$\notin$	$\notin$	$\notin$	$\notin$
$\notin$	$\notin$	$\notin$	$\notin$	$\notin$	$\notin$	$\notin$	$\notin$

From (5) and (8) we get $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Hence, we have proved

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

This method of proof is called **Proof by Exhaustion.**

EXERCISE 3.3

Using membership table prove each of the following :

1. $(A \cap B)' = A' \cup B'$
2. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
3. $A - (B \cap C) = (A - B) \cup (A - C)$

3.5 PROBLEMS BASED ON 3 SETS

We have learnt the formula.

$$n(A \cup B) = n(A) + n(A) - n(A \cap B) \quad \text{if } A \cap B \neq \{\ \}$$

$$n(A \cup B) = n(A) + n(B) \quad \text{if } A \cap B = \{\ \}$$

Let us extend this formula for 3 sets.

$$n[A \cup B \cup C] = n[A \cup (B \cup C)]$$

$$= n(A) + n(B \cup C) - n[A \cap (B \cup C)]$$

$$= n(A) + n(B \cup C) - n[(A \cap B) \cup (A \cap C)]$$

$$= n(A)+n(B)+n(C)-n(B\cap C)-\left[n(A\cap B)+n(A\cap C) - n[(A\cap B)\cap(A\cap C)]\right]$$

$$= n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$$

$$= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)]$$

We can solve many problems using this formula.

Example 3.9 : A tooth-paste company interviewed 141 people in a city. It was foundout that 90 use Brand A paste, 80 use Brand B paste, 50 use Brand C paste, 40 use both A and B, 28 use both B and C, 26 use both C and A, and 15 use all these three pastes. Find how many use (i) A and B and not C (ii) B only and (iii) C and A and not B.

Solution : Venn diagram method : Let us represent these three Brands in a Venn diagram.

Region e represents 15

$$b + e = 40$$

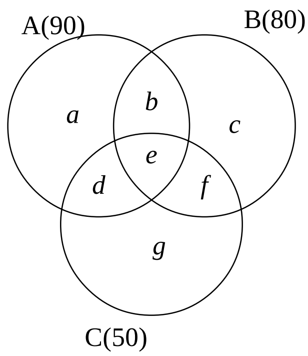
$$\therefore b = 40 - 15 = 25$$

$$e + f = 28$$

$$\therefore f = 13$$

$$d + e = 26$$

$$\therefore d = 11$$



Now $a + b + d + e = 90$

$$a + 25 + 11 + 15 = 90$$

$$\therefore a = 39$$

$$b + c + e + f = 80$$

$25 + c + 15 + 13 = 80$

$c = 27$

Also $d + e + g + f = 50$

$11 + 15 + g + 13 = 50$

$g = 11$

$\therefore\ a = 39,\ b = 25,\ c = 27,\ d = 11,\ e = 15,\ f = 13,\ g = 11.$

(i) Number of people who are A and B are not C is the region represented by b.

$\therefore$ 25 use A and B and not C.

(ii) Number of people who use only B is the region represented by c.

$\therefore$ 27 use B only.

(iii) Number of people who use C and A and not B is the region represented by d

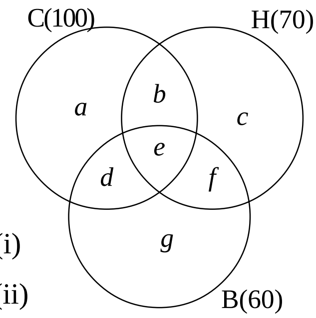
$\therefore$ 11 use C and A and not B.

For this type of problem, it is easy to use Venn diagram method.

Example 3.10 : In a city, it is found that 100 play Cricket, 70 play Hockey, 60 play Basket-ball. 41 play Cricket and Hockey, 33 play Basket-ball and Hockey, 27 play Basket ball and Cricket. In total 140 play either one of these three games. Find the number of people who play all these three games.

Solution : Method 1.

Venn diagram method



The region e represents all the three games. Let the number of people who play all these three games be x.

Now, we hav $a + b + d + e = 100$ ... (i)

$b + c + e + f = 70$ ... (ii)

$d + e + f + g = 60$ ... (iii)

$b + e = 41$ ... (iv) $\therefore\ b = 41 - x$

$e + f = 33$ ... (v) $f = 33 - x$

$d + e = 27$ ... (vi) $d = 27 - x.$

$a + 41 - x + 27 - x + x = 100$ ... (i)

$a = 32 + x$

$$41 - x + c + x + 33 - x = 70 \quad \text{... (ii)}$$

$$c = -4 + x$$

$$27 - x + x + 33 - x + g = 60$$

$$g = x$$

We know $a + b + c + d + e + f + g = 140$

$$32 + x + 41 - x - 4 + x + 27 - x + x + 33 - x + x = 140$$

$$129 + x = 140$$

$$x = 11$$

$\therefore$ 11 play all these three games.

Method 2 : Formula method

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

$$140 = 100 + 70 + 60 - 41 - 33 - 27 + x$$

$$140 = 129 + x$$

$$\therefore x = 140 - 129 = 11$$

$\therefore$ 11 play all these three games.

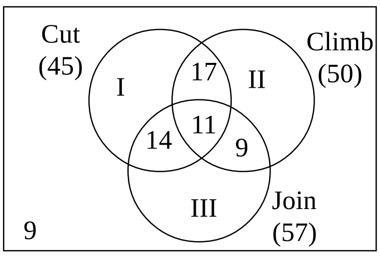
Here, the formula method is easier.

Example 3.11 : Aviral is a section chief for an electric utility company. The employees in his section cut down trees, climb poles and join wire. He reported the following information to the management of the utility.

''Of the 100 employees in my section. 45 can cut trees ; 50 can climb poles; 57 can join wire ; 28 can cut trees and climb poles ; 20 can climb poles and join wire ; 25 can cut trees and join wire ; 11 can do all the three and 9 can't do any of these three''.

The management decided to punish him. Why ?

Solution : Let us represent these data in a Venn diagram.



Region I : $45 - (17 + 11 + 14) = 3$

Region II : $50 - (17 + 11 + 9) = 13$

Region III : $57 - (14 + 11 + 9) = 23$

Total employees according to the given data

$$= (3 + 17 + 13 + 14 + 11 + 9 + 23) + 9 = 99$$

But there are 100 employees. So Aviral has given false information. That is why the management has decided to punish him.

EXERCISE 3.4

1. Human blood is classified (typed) according to the presence or absence of the specific antigens A, B and Rh in red blood cells. Blood is called A-positive if the individual has the A and Rh, but not the B antigen. A person having only the A and B antigens is said to have AB negative blood. A person having only the Rh antigen has type O positive blood. A person having only the A antigen is said to have A negative blood. A person having only B antigen is said to have B negative blood. A person having only B and Rh is B positive. A person having all these three is said to have AB positive. A person who is not having these three antigens is said to have O negative blood. Identify the blood types of the individuals and sketch a Venn diagram.

2. Using Venn diagram solve.

   In a certain hospital, the following data were recorded. 25 patients had the A antigen; 17 had the A and B antigens ; 27 had the B antigen ; 22 had the B and Rh antigens; 30 had the Rh antigen ; 12 had none of the antigens ; 16 had the A and Rh antigens; 15 had all three antigens. How many patients (a) were represented ? (b) has exactly one antigen ? (c) had exactly two antigens ? (d) had O positive blood ? (e) had AB positive blood ? (f) had B negative blood ? (g) had O negative blood ? (h) had A positive blood ? [Use the Venn diagram obtained in problem (1)]

3. A number of people were asked whether they liked drinks of orange, lemon or grape flavour. The replies are 85 liked orange, 65 liked grape ; 90 liked lemon ; 30 liked orange and grape ; 45 liked orange and lemon ; 40 liked lemon and grape ; 15 liked lemon, orange and grape. 25 liked none of the three. Find the total number of people interviewed and the number who liked orange alone ; lemon alone ; and grape alone.

4. In a group of girls, 20 play volley ball, 21 play badminton, and 18 play table-tennis ; 7 play volleyball only, 9 play badminton only ; 6 play volleyball and badminton only and 2 play badminton and table-tennis only. (i) How many play all the three games ? (ii) How many play volleyball and table tennis only ? (iii) How many play table-tennis only ? (iv) How many girls are there altogether?

5. After a genetics experiment, the number of pea plants having certain characteristics was tallied, with the following results.

   22 were tall ; 25 had green peas ; 39 had smooth peas ; 9 were tall and had green peas; 17 were tall and had smooth peas ; 20 had green peas and smooth peas ; 6 had all three characteristics ; 4 had none of the characteristics. (a) Find the total number

of plants counted. (b) How many plants were tall and had peas that were neither smooth nor green ? (c) How many plants were not tall but had peas that were smooth and green ?

6. A survey of 63 business people found that ; 30 had desktop computers ; 22 had laptop computers ; 39 had fax machines ; 15 had desktop computers and laptop computers; 18 had a desktop computer and a fax machine ; 14 had a laptop computer and a fax machine ; 12 had all the three.

   (a) How many had only laptop computers ?

   (b) How many had only fax machines ?

   (c) How many had a fax machine and laptop but not a desktop computer ?

   (d) How many had a fax machine or a laptop, but not a desktop computer ?

   (e) How many had none of the three items ?

7. A survey of 500 farmers in a state showed the following : 125 grew only wheat ; 110 grew only corn ; 90 grew only ragi ; 200 grew wheat ; 60 grew wheat and corn ; 50 grew wheat and ragi, 180 grew corn. Find the number who (a) grew exactly one of the three (b) grew all the three (c) did not grew any of the three (d) grew exactly two of the three.

8. In a bird sanctuary, 41 different species of birds are being studied. Three large trad feeders are constructed, each providing a different type of bird feed. One feeder has sunflower seeds. A second feeder has a mixture of seeds and the third feeder has small pieces of fruit. The following information was obtained. 20 species ate sunflower seeds ; 22 species ate the mixture, 11 species ate the fruit ; 10 species ate the sunflower seeds and the mixture ; 4 species ate the sunflower seeds and the fruit ; 3 species ate the mixture and the fruit ; 1 species ate all the three. How many species ate (a) none of the foods ? (b) the sunflower seeds, but neither of the other two foods ? (c) the mixture and the fruit, but not the sunflower seeds ? (d) the mixture or the fruit, but not the sunflower seeds ? (e) exactly one of the foods ?

9. 33 districts were surveyed to determine whether they had a cricket team, a hockey team or a volley ball team. The following information was determined. 16 had cricket, 17 had hockey, 15 had volley ball, 11 had cricket and hockey, 7 had cricket and volley ball, 9 had hockey and volley ball, 5 had all the three teams. How many had only (a) a hockey team ? (b) cricket and hockey but not volley ball team ? (c) cricket or hockey team ? (d) cricket or hockey but not volley ball ? (e) exactly two teams ?

10. An inquiry has been carried out into the popularity of three brands of tooth-paste A, B and C 5951 people interviewed : (i) 671 did not use tooth-paste (ii) 2480 used brand A (iii) 800 used brand C only (iv) 1340 used brand B only (v) only 400 used both B and C but not A. Verify the data.

3.6 RELATIONS AND FUNCTIONS

Consider the two non-empty sets A and B. Let $x \in$ A and $y \in$ B. Then we call (x, y) as an ordered pair. The set of all ordered pairs (x, y) where $x \in$ A and $y \in$ B is called the **Cartesian product** of A and B.

It is denoted by $A \times B = \{(x, y) / x \in A \text{ and } y \in B\}$.

Example 3.12 : Given A = {1, 2, 3}, B = {4, 5} find A × B.

Solution : A × B = {(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)}

Any subset of A × B is called a **relation** from A to B. Let us name the subset by R from A to B. If any ordered pair $(x, y) \in$ R, we write x R y. We read it as 'x is related to y'.

Example 3.13 : Given A = {1, 2, 4, 8}. Write down the set of ordered pairs having the relation 'is the divisor of''.

Solution : We know 1 is the divisor of 1, 2, 4, 8.2 is the divisor of 2, 2 is the divisor of 4, 2 is the divisor of 8.4 is the divisor of 4, 4 is the divisor of 8.8 is the divisor of 8.

Hence, R = {(1,1), (1,2), (1,4), (1,8), (2,2), (2,4), (2,8), (4,4), (4,8), (8,8)}

Let us consider a special case of relation. Let the cost of 1 apple be Rs. 8. Let us indicate this relation in a table of values.

Number of apples	Cost (Rs.)
1	8
2	16
3	24
4	32
5	40
:	:
.	.

In general, the cost for buying n apples will be 8 times the number of apples or Rs. 8n. We can represent the cost, C of n times by the equation $C = 8n$.

Since, the value depends on the value of n, we refer to n as the independent variable and C as the dependent variable. Note that for each value of the independent variable, n, there is one and only value of the dependent variable C. Such an equation is called a **function.**

In the equation $C = 8n$, the value of C depends on the value of n, so we say that ''C is a function of n''.

Is the equation $y = 3x - 2$ a function ? To answer this question, we must ask, "Does each value of x corresponds to a unique value of y".

The answer is yes : therefore, this equation is a function.

For the equation $y = 3x - 2$, we say that "y is a function of x" and write $y = f(x)$. The notation $f(x)$ is read as f of x". When we are given an equation that is a function, we may replace the y in the equation with $f(x)$, since $f(x)$ represents y. Then $y = 3x - 2$ may be written $f(x) = 3x - 2$.

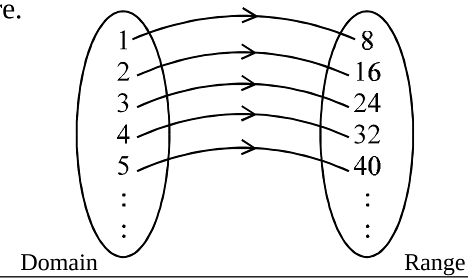
Functional Notation : Let a function f be defined on the set of natural numbers as $f = \{(1, 3), (2, 4), (3, 5), (4, 6), (5, 7), ... \}$, then this can be defined by $f : \mathrm{N} \rightarrow \mathrm{N}$, $f(x) = x + 2$.

A function is a special type of relation where each value of the independent variable corresponds to a unique value of the dependent variable.

Domain : The set of values that can be used for the independent variable is called the **domain** of the function.

Range : The resulting values obtained for the dependent variable is called the range.

The domain and range for the function $C = 8n$; $n \in \mathrm{N}$ are illustrated in the following figure.



Definition : *Let A and B be two non-empty sets. A function f from set A into set B is a subset of A × B such that for each and every element of A there corresponds one and only element of B. We denote this as $f : A \rightarrow B$*

The domain of the function f is the set A

The codomain of the function f is the set B

The range of f is either set B or a subset of B.

Let us consider the two sets A and B. A = {1, 2, 3} and B = {1, 4, 9, 16}.

Let $f : A \to B$ defined by $f(x) = x^2$

Then $f = \{(1, 1), (2, 4), (3, 9)\}$

Here Domain $= A = \{1, 2, 3\}$

Co-domain $= B = \{1, 4, 9, 16\}$ and range $= \{1, 4, 9\}$

Image and Pre-image

The image of 1 is 1, the image of 2 is 4 and the image of 3 is 9.

The pre-image of 1 is 1, the pre-image of 4 is 2 and the pre-image of 9 is 3.

3.6.1 Representation of a function

We can represent a function in any one of the following forms.

(1) An arrow diagram

(2) A set of ordered pairs

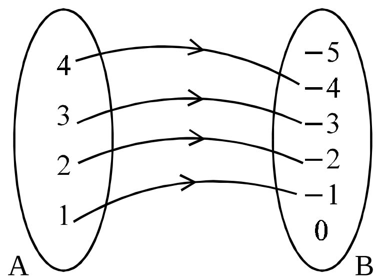
(3) A table

(4) A graph.

(1) Representation by an arrow diagram

We draw any two closed curves to represent the set A and set B and enter the elements inside. We indicate by arrow which element of B is associated with each one of the elements in A.

Let $A = \{4, 3, 2, 1\}$, $B = \{-5, -4, -3, -2, -1, 0\}$



Here, $A = \{4, 3, 2, 1\}$, is the **Domain.**

$B = \{-5, -4, -3, -2, -1, 0\}$ is the **Co-domain.**

$R = \{-4, -3, -2, -1\}$ is the Range.

We denote this function as

$f : A \rightarrow B$, where A = {4, 3, 2, 1} and B = {–5, –4, –3, –2, –1, 0}

defined by $f(x) = -x$

(2) Representation by an ordered pair

Consider the sets P and Q, where P = {1, 2, 3} and Q = {2, 3, 4, 5}

Let $f : P \rightarrow Q$ defined by $f(x) = x + 1$.

then $f = \{(1, 2), (2, 3), (3, 4)\}$

In general, we can represent a function,

$f : A \rightarrow B$ as $f = \{(x, y) : x \in A, y \in B\}$

(3) Representation by using Table

Here, we present the function in a Tabular form. Consider a function from A to B.

Let $f(x) = 2x$

A = {–3, –4, –5}, B = {–6, –8, –10, –12}

x	–3	–4	–5
$f(x)$	–6	–8	–10

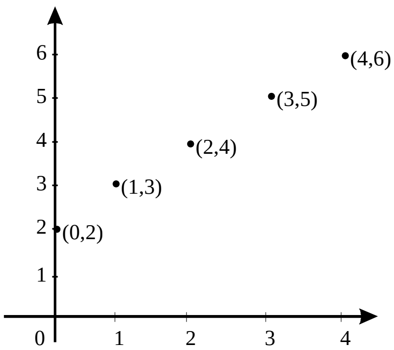
(4) Representation by a graph

Consider the sets A and B, given A = {0, 1, 2, 3, 4} and B = {2, 3, 4, 5, 6}

$f : A \rightarrow B$ is defined as $f(x) = x + 2$

The set of ordered pairs of this function is f={(0,2), (1,3), (2,4), (3,5), (4,6)}

We can represent the first elements of the ordered pair on the X-axis of a graph sheet and the values of the second elements on the Y-axis. Let us plot the points on the graph sheet.



Note : In the above graphical representation, we cannot join the points in the graph sheet as A is a subset of W.

The following points should be remembered in a function.

(a) Each element of set A is mapped (is assigned to) on to a unique element in set B.

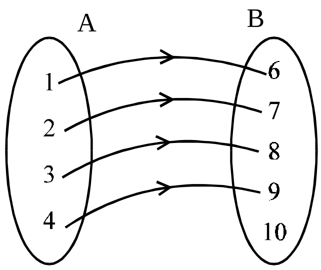
(b) Two or more elements of set A can however be assigned to the same element of set B.

(c) There may be a few elements in set B which are not assigned by any element of set A.

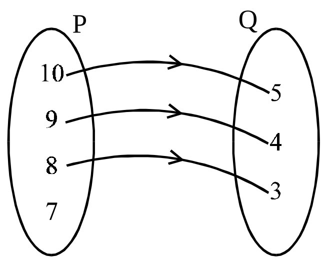
(d) The range of a function is always a subset of the co-domain of the function. It may be proper subset or an improper subset.

Example 3.14 : Which of the following are examples for function.

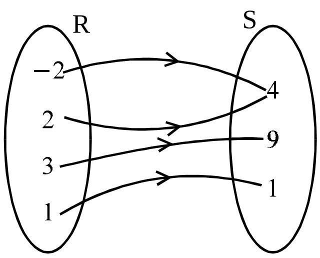
(*i*)



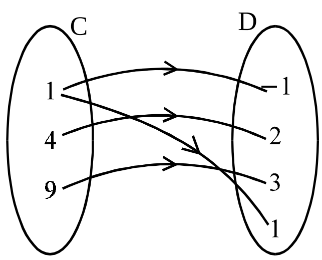
(*ii*)



(*iii*)



(*iv*)



Solution : (i) Every element in A is mapped on to a unique element in B. Hence it is a function.

(ii) The element 7 in P does not have a mapping in Q. Hence, it is not a function.

(iii) Every element in R is mapped onto a unique element in S. Hence, it is a function.

(iv) The element 1 in C is mapped onto two elements in D, namely –1 and 1. In other words 1 is having two images. Hence, it is not a function.

Example 3.15 : Consider the relation R = {(1, 1), (4, 2), (9, 3), (4, –2)} state with reason whether this is a function or not.

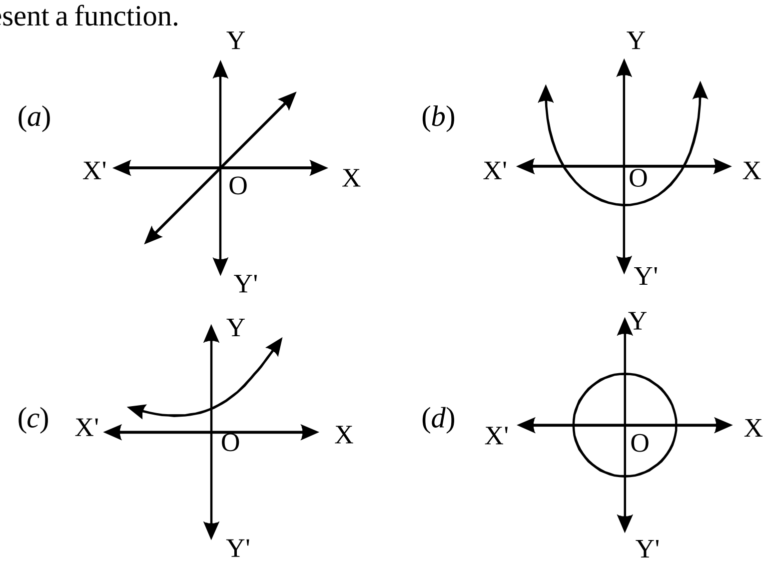
Solution : It is not a function as the first element of the ordered pairs gets repeated twice (4 comes twice). i.e., 4 has two images.

Suppose, we are given a graph. We can determine whether the given graph represents a function by using the **vertical line test.**

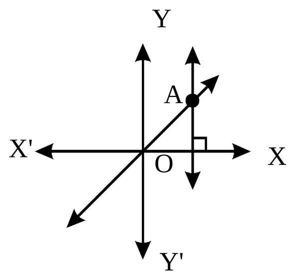
If a vertical line can be drawn so that it intersects the graph at more than one point, then each x does not have a unique y and the graph **does not** represent a function.

If a vertical line cannot be made to intersect the graph in at least two differenet points, then the graph represents a function.

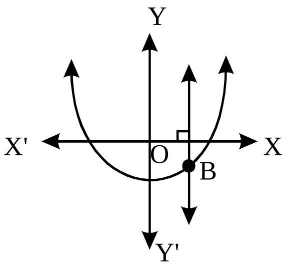
Example 3.16 **:** Use the vertical line test to determine which of the graph represent a function.



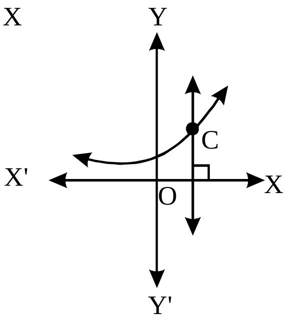
Solution : (a) This graph represents a function as the vertical line intersect the graph at only one point (say A)



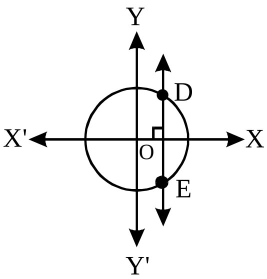
(b) This graph represents a function as the vertical line intersect the graph at only one point (say B)



(c) This graph represents a function as a the vertical line intersects the graph at only one point (say C)



(d) This graph does not represents a function since the vertical line cuts the graph at two points. (Say D and E)



3.6.2 Classification of functions

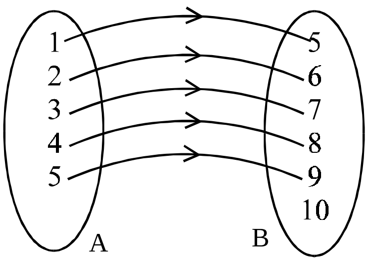
Functions can be classified (a) one-one function (b) many-one function (c) into function (d) onto function (d) one-one into function (f) one-one onto function (g) many-one into function (h) many-one onto function (i) constant function and (j) identity function.

(a) One-one function : One-one function means each and every element in domain A correspond uniquely to different elements in codomain B. That is, no element of B is the image of more than one element in set A.

The function $f : A \to B$ is one-one if different elements in A have different images in B.

Example : Let $f : A \to B$ where A = {1, 2, 3, 4, 5} and B = {5, 6, 7, 8, 9, 10} defined by $f(x) = x + 4$.

We have

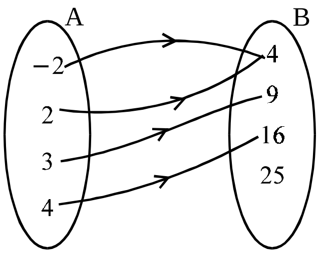


Here, different elements of A have different images in B and so this is one-one function.

(b) Many-one function : When two or more elements of the domain A correspond to the same element of the co-domain B, then it is called a many-one function.

Example : Let f : A $\rightarrow$ B where A = {–2, 2, 3, 4} and B = {4, 9, 16, 25} defined by $f(x) = x^2$. The elements –2 and 2 in the domain A are mapped onto the same element 4 in the co-domain B. $\therefore$ f is a many-one function.

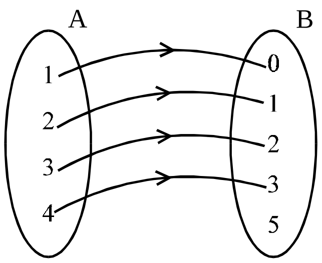
We have



(c) Into function : Those functions whose range is not the whole co-domain B, that is, certain elements of co-domain B are left unused, are called ''into functions''.

Example : Let f : A $\rightarrow$ B where A = {1, 2, 3, 4} and B = {0, 1, 2, 3, 5} defined by $f(x) = x - 1$.

We have



In the co-domain B the element –5 does not have a pre-image

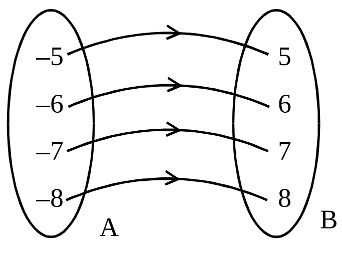
$\therefore$ f is into function.

(d) Onto function : Those functions whose range is equal to its co-domain B, that is, no element of co-domain B is left unused, are called onto function.

In short, A function f : A $\rightarrow$ B is said to be an onto function if its range is equal to its codomain B.

Example : Let f : A $\rightarrow$ B where A = {–5, –6, –7, –8} and B = {5, 6, 7, 8} defined by $f(x) = -x$.

We have



In the above function the range is equal to its co-domain.

B = {5, 6, 7, 8} and Range R = {5, 6, 7, 8}

B = R. ∴ f is an onto function.

(e) One-one into function : It is a function which is both one-one and into.

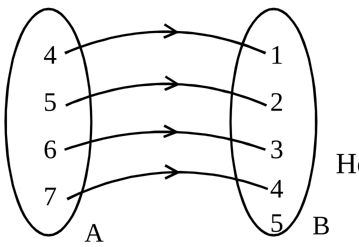
Such functions, therefore, have the following characteristics.

(i) No two elements of the domain corresponds to the same element of the co-domain.

(ii) There is atleast one element of the co-domain which does not correspond to any element of the domain.

Example : Let A = {4, 5, 6, 7} and B = {1, 2, 3, 4, 5}

$f : A \rightarrow B$ defined by $f(x) = x - 3$.



Here f is one-one into.

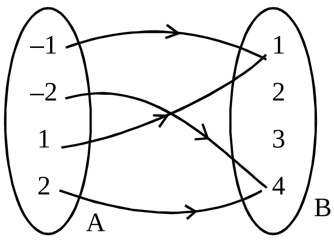
(g) Many-one into function : It is a function which is both many-one and into. These functions have the following characteristics.

(i) There are at least two elements of the domain which correspond to the same element of the codomain.

(ii) There is atleast one element of the codomain which does not correspond to any element of the domain.

Example : Let A = {–1, –2, 1, 2} and B = {1, 2, 3, 4}, $f : A \rightarrow B$ defined by $f(x) = x^2$.

We have



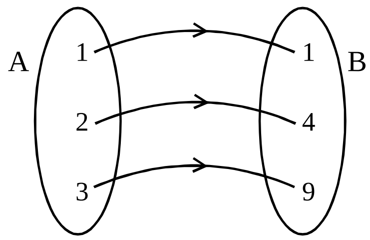
Here f is many-one into.

(f) One-one onto function : It is a function which is both one-one and onto. Such functions have the following characteristics.

(i) No two elements of the domain correspond to the same element of the codomain.

(ii) Every element of the codomain corresponds to the some element of the domain.

Example : Let A = {1, 2, 3} and B = {1, 4, 9}. f : A $\to$ B defined by $f(x) = x^2$.



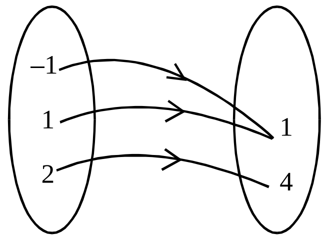
f is a one-one onto function.

(h) Many-one onto function : It is a function which is both many-one and onto. Such functions have the following characteristics.

(i) There are atleast two elements of the domain, which correspond to the same element of the co-domain.

(ii) Every element of the co-domain corresponds to some elements of the domain.

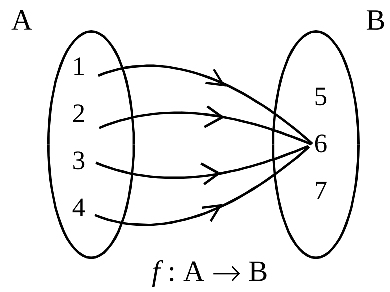
Example :



Here f is many-one into.

(i) Constant function : A function f : A $\to$ B is called a constant function if every element of A has the same image in B. The range of the function is a singleton. It is also a many-one function.

Examples : Let f : A $\to$ B, where A = {1, 2, 3, 4} and B = {5, 6, 7}, defined by $f(x) = 6$.



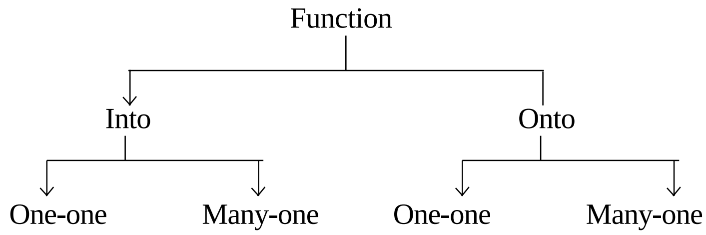
f : A $\to$ B

(j) Identity function : Let A be a non-empty set. A function $f : A \to A$ is called an identity function if each element of A is associated with itself under F.

i.e., $f(x) = x$ for all $x \in A$

Example : Let A = {1, 2, 3, 4}, $f : A \to A$ defined by $f(x) = x$ is an identity function. Here $f(1) = 1$, $f(2) = 2$, $f(3) = 3$, $f(4) = 4$.

Thus the above classification can be exhibited by the tree-diagram.



Example 3.17 : (i) Let A = {3, 6, 9, 12}, B = {1, 2, 3, 4, 5, 6} and $f : A \to B$ is defined by $f(x) = \frac{1}{3}x + 1$. Represent f as (a) an arrow diagram (b) a set of ordered pairs (c) a table and (d) a graph.

Solution : $f(x) = \frac{1}{3}x + 1$

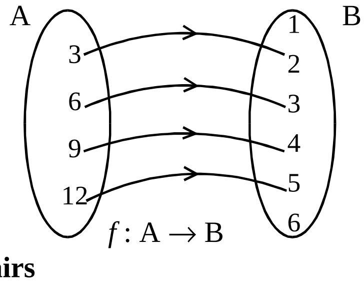
$$f(3) = \frac{1}{3}(3) + 1 = 2$$

$$f(6) = \frac{1}{3}(6) + 1 = 3$$

$$f(9) = \frac{1}{3}(9) + 1 = 4$$

$$f(12) = \frac{1}{3}(12) + 1 = 5$$

(a) Arrow diagram



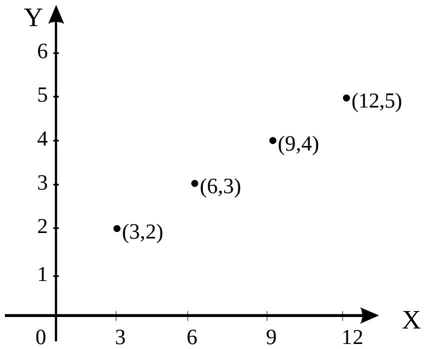
(b) Set of ordered pairs

$f = \{(3, 2), (6, 3), (9, 4), (12, 5)\}$

(c) Tabular form :

x	3	6	9	12
$f(x)$	2	3	4	5

(d) Graph :



Example 3.18 : If $f : R \to R$ (where R denotes the set of real numbers) is defined by

$$f(x) = \begin{cases} 4x - 3 & \text{if} \quad x > 2 \\ x^2 + x - 2 & \text{if} \quad -1 < x \leq 2 \\ 3x + 1 & \text{if} \quad x \leq -1 \end{cases}$$

find (i) $f(3)$, (ii) $f(2)$, (iii) $f(-2)$

Solution : (i) $3 > 2$. Hence, we have to use $f(x) = 4x - 3$

$$\therefore f(3) = 4(3) - 3 = 9$$

(ii) 2 is in the range of $-1 < x \leq 2$. Hence, we have to use

$$f(x) = x^2 + x - 2.$$

$$\therefore f(2) = 2^2 + 2 - 2 = 4$$

(iii) $-2 < -1$. Hence, we have to use $f(x) = 3x + 1$.

$$\therefore f(-2) = 3(-2) + 1 = -5.$$

(i) $f(3) = 9$ (ii) $f(2) = 4$, (iii) $f(-2) = -5$

Example 3.19 : **Given :** $f = Z \to N$ is defined by $f(x) = x + 1$ test whether this represents a function or not. Give reason.

Let $x = -1$ $(-1 \in Z)$

(Z is the set of integers, N is the set of natural numbers)

$f(-1) = -1 + 1 = 0, \qquad 0 \notin N$

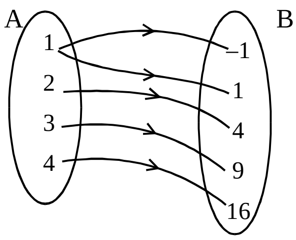
There is no image for –1.

Hence, it is not a function.

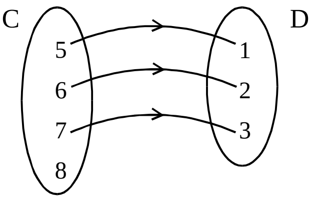
EXERCISE - 3.5

1. Which of the following are functions and not functions. State the reason.

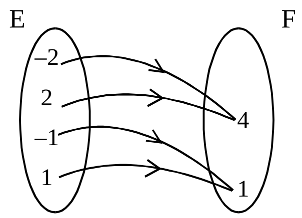
(i)



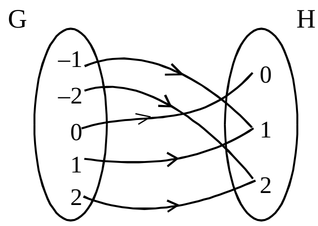
(ii)



(iii)



(iv)



(v) R = {(1, 2), (2, 3), (3, 4), (4, 5), (1, 6)}

(vi) S = {(–1, 1), (1, 1), (–2, 2), (2, 2)}

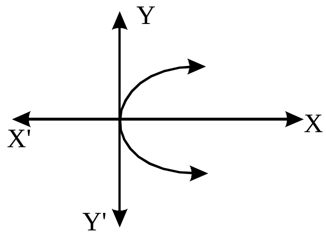
(vii)

x	–3	3	4	–4
y	9	9	16	16

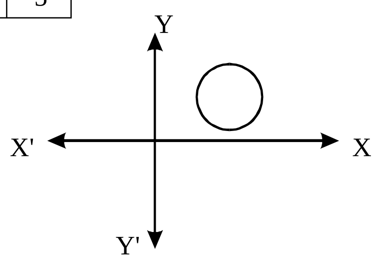
(viii)

x	1	2	3	4
y	2	3	4	5

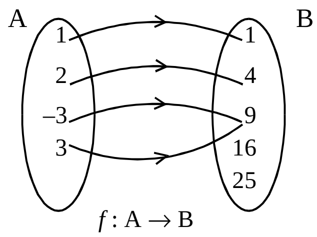
(ix)



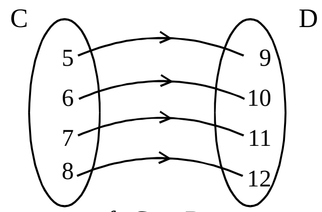
(x)



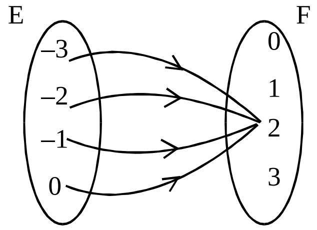
2. Find whether the following functions are one-one into, many-one into, one-one onto or many-one onto.

(a) 

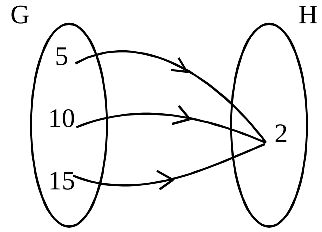
$f : A \to B$

(b) 

$f : C \to D$

(c) 

$f : E \to F$

(d) 

$f : G \to H$

(e) Let A = {7, 8, 9, 10, 11}, B = {4, 5, 6}

(i) R = {(7, 4), (8, 4), (9, 5), (10, 4), (11, 5)}

(ii) S = {(7, 4), (8, 4), (9, 4), (10, 4), (11, 4)}

3. If {(–6, a), (b, 4), (–2, c), (d, 7)} is an identity function find the values of a, b, c and d.

4. Find the domain and range.

(i) M = {(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)}

(ii) R = {(–2, 4), (–1, 1), (2, 4), (1, 1), (–3, 9)}

5. Find the pre-image(s) of 3 in the functions.

P = {(–1, 1), (–2, 2), (–3, 3), (3, 3), (4, 4)}

6. Find the pre-image(s) of 4 in the function.

Q = {(1, 4), (2, 4), (3, 4), (4, 4)}

7. Represent each of the following functions as

(a) an arrow diagram, (b) a set of ordered pairs, (c) a table and (d) as a graph.

(i) If A= {–3, –1, 1, 3}, B = {0, 1, 2, 3, 4} and $f : A \to B$ is defined by

$$f(x) = \frac{3-x}{2}$$

(ii) If P = {3, 4, 5, 6, 8}, Q = Set of real numbers and $f : P \to Q$ is defined by

$$f(x) = \frac{12}{x-2}$$

8. If A = {0, 1, 2, 3}, B = {5, 3, 1, –1, –3, –5}, R = {(*x*, *y*) : $y = 3 - 2x$, $x \in$ A, $y \in$ B} (i) List the elements of R (ii) What is the co-domain ? (iii) What is the range of R (iv) Identify the function.

9. Given : P = {–2, –1, 0, 1}, Q = {1, –2, 6, –3}

R = {(*x*, *y*) : $y = (x^2 - 3)$, $x \in$ P, $y \in$ Q}

(i) List the elements of R, (ii) What is the domain of R. (iii) What is the range of R, (iv) Is the relation a function ? If so, identify the function.

10. Given :

$$f(x) = \begin{cases} 3x-2, & -5 \le x < 0 \\ x+3, & 0 \le x < 5 \\ 2x-3, & 5 \le x < 10 \end{cases} \quad \text{find } \frac{2f(-3) + f(2)}{f(7) - f(-1)}$$

11. Given :

$$f(x) = \begin{cases} 2x^2 - 3, & x > 4 \\ 7x + 4, & -3 \le x < 4 \\ 4x - 3, & x \le -4 \end{cases} \quad \text{find } \frac{f(-2) - f(5)}{f(-5)}$$

3.7 COMPOSITION OF FUNCTIONS

Composition of functions is the combination of functions to arrive at the final range for a given domain. This operation helps us to find directly the range instead of undergoing functional operations.

If we have two functions *f* and *g* then the **composite function** $f \circ g$ [read as *f* composite *g*] to mean that we shall obtain the results of mapping of *g* first and then carry out the mapping *f* on these results second.

Example : *g* is the mapping $x \to x + 2$

f is the mapping $x \to x^2$.

Then $f \circ g$ is the mapping $x \to (x + 2)^2$ but $g \circ f$ is the mapping $x \to x^2 + 2$. Thus $f \circ g$ is not the same relation $g \circ f$.

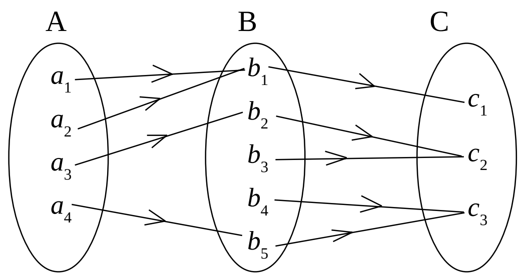
One or two numerical values will probably help in understanding the above.

Suppose $x = 7$, then $g : 7 \to 9$; $f : 7 \to 49$

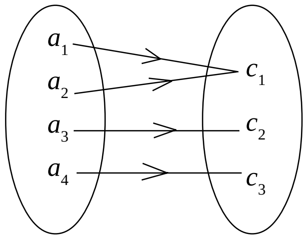
and $f : 9 \to 81$; $g : 49 \to 51$

$f \circ g : 7 \to 81$ and $g \circ f : 7 \to 51$

We shall now look at a composite function in diagram form.



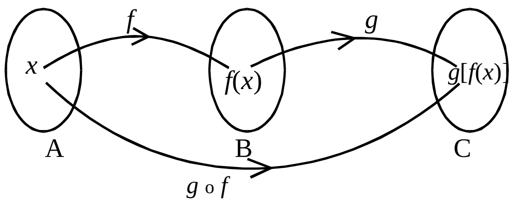
$f : \mathrm{A} \to \mathrm{B}$ $\qquad$ $g : \mathrm{B} \to \mathrm{C}$



$g \circ f : \mathrm{A} \to \mathrm{C}$

Is the compoition of the two functions also a function ? Yes. It can thus be seen that we are often able to construct new functions by the composition of other functions.

Definition : Let A, B, C be three non-empty sets. Let $f : \mathrm{A} \to \mathrm{B}$, $g : \mathrm{B} \to \mathrm{C}$ be two functions. Here we have taken the domain of g to be co-domain of f. Define a function $g \circ f : \mathrm{A} \to \mathrm{C}$ as $(g \circ f)(x) = g[f(x)]$ for all $x \in \mathrm{A}$. Since $f(x) \in \mathrm{B}$, $g[f(x)] \in \mathrm{C}$. The function $g \circ f$ so obtained is called the composition of f and g (read as g circle f). The function $g \circ f$ can be represented by the diagram as shown in the figure.



Examples : 3.20. Given $f(x) = 3x - 2$, $g(x) = 2x^2$. Find $f \circ g$ and $g \circ f$ what do you find ?

(i) $\qquad (f \circ g)\, x = f[g(x)] = f(2x^2)$

$= 3(2x^2) - 2 = 6x^2 - 2 \qquad$ [Since $f(x) = 3x - 2$]

(ii) $(g \circ f)x = g[f(x)]$

$= g[3x - 2]$

$= 2(3x - 2)^2$ [Since $g(x) = 2x^2$]

$= \mathbf{18x^2 - 24x + 8}$

We find that $f \circ g \neq g \circ f$. That is, composition of functions is not commutative.

Example 3.21 : If $f : R \rightarrow R$ is defined by $f(x) = ax + 3$ and $g : R \rightarrow R$ is defined by $g(x) = 4x - 3$ find 'a' so that $f \circ g = g \circ f$.

$(f \circ g)(x) = f[g(x)] = f[4x - 3] = a(4x - 3) + 3 = 4ax - 3a + 3$

$(g \circ f)(x) = g[f(x)] = g[ax + 3] = 4(ax + 3) - 3 = 4ax + 12 - 3$

$= 4ax + 9$

It is given, $f \circ g = g \circ f$

$\therefore \quad 4ax - 3a + 3 = 4ax + 9$

$-3a = 6$

$\therefore a = -2$

Example 3.22. Given $f(x) = 5x + 2$, $g(x) = 2x - 3$, $h(x) = 3x + 1$, verify $f \circ (g \circ h) = (f \circ g) \circ h$.

LHS. $= f \circ (g \circ h)$

$(g \circ h)(x) = g[h(x)]$

$= g[3x + 1]$

$= 2(3x + 1) - 3$

$= 6x - 1$ [say $k(x)$]

$[f \circ (g \circ h)](x) = (f \circ k)(x)$

$= f[k(x)]$

$= f[6x - 1]$

$= 5(6x - 1) + 2$

$\therefore [f \circ (g \circ h)](x) = 30x - 3$ ... (1)

RHS : $(f \circ g) \circ h$

$(f \circ g)(x) = f[g(x)]$

$= f(2x - 3)$

$= 5\ (2x - 3) + 2$

$= 10x - 13$ [say $l\ (x)$]

$\therefore\ [(f \circ g) \circ h]\ (x) = (l \circ h)\ (x)$

$= l\ [h\ (x)]$

$= l\ [3x + 1]$

$= 10\ (3x + 1) - 13$

$\therefore\ [(f \circ g) \circ h]\ (x) = 30x - 3$ ... (2)

From (1) and (2)

$\therefore\ [f \circ (g \circ h)]\ (x) = [(f \circ g) \circ h]\ x$

Since x is arbitrary

$f \circ (g \circ h) = (f \circ g) \circ h$

Hence, verified.

Note : (1) We have seen $f \circ g \neq g \circ f$.

That is composition of functions is **not commutative.**

(2) We have also seen $f \circ (g \circ h) = (f \circ g) \circ h$

That is, composition of functions is always Associative.

EXERCISE 3.6

1. Given $f\ (x) = 3x - 7$ and $g\ (x) = 2 - 3x$. Find $f \circ g$ and $g \circ f$.
2. If $f\ (x) = 2x^2$, $g(x) = 3x - 1$, find $f \circ g$ and $g \circ f$
3. Given : $f\ (x) = 7x - 3$ and $g\ (x) = x^2 - 2$. Show that $f \circ g \neq g \circ f$.
4. Given : $f\ (x) = 3x^2 + 2x - 1$ and $g\ (x) = x - 1$. Show that $g \circ f \neq f \circ g$.
5. Given : $f\ (x) = 3x - 2$, $g\ (x) = kx + 3$, find k so that $f \circ g = g \circ f$.
6. Given : $f\ (x) = 3 + x$, $g\ (x) = x^2$, $h\ (x) = \frac{1}{x}$, show that the composition of functions is associative.
8. Given : f $(x) = x^2 + 4$, g $(x) = 3x - 2$, $h\ (x) = x - 5$ show that the composition of functions is associative.
9. Given $f\ (x) = x^2 - 1$, $g\ (x) = x + 1$, $h\ (x) = 1 - x$, verify that $h \circ (g \circ f) = (h \circ g) \circ f$.
10. Given $f\ (x) = x - 2$, $g\ (x) = 3x + 5$, $h\ (x) = 2x - 3$, verify that $(g \circ h) \circ f = g \circ (h \circ f)$.

4. CONSUMER ARITHMETIC

4.1 Difference between simple interest and compound interest

4.1.1. Introduction

We have learnt about simple interest and compound interest. Let us recall.

In the case of simple interest, the interest is charged on the original sum borrowed or deposited (Principal) throughout the period of loan or deposit.

In the case of compound interest, the principal for the second term is the sum of the principal for the first term and the simple interest for the first term. Similarly, the principal for the third term is the sum of the principal for the second term and simple interest for the second term and so on.

The formula for Calculating Amount is $\boxed{A = P\left(1+\frac{r}{100}\right)^n}$ where P is the Principal, r is the rate per cent annually (compound interest rate), and n is the number of terms.

The formula for finding compound interest is $\boxed{C.I. = P\left(1+\frac{r}{100}\right)^n - P.}$

If we want to invest our money, we have to decide whether to invest in compound interest scheme or simple interest scheme. So, there arises a comparison. That is, we have to find the difference between the compound interest and the simple interest.

4.1.2 Formula to calculate the difference between Compound Interest and Simple Interest.

Let P be the principal, r be the rate percent annually and n = 2 years

$$C.I = P\left(1+\frac{r}{100}\right)^2 - P$$

$$= P\left[1+\frac{2r}{100}+\frac{r^2}{100^2}\right] - P$$

$$= P + \frac{2Pr}{100} + \frac{Pr^2}{100^2} - P$$

$$\text{C.I.} = \frac{2Pr}{100} + \frac{Pr^2}{100^2}$$

$$\text{S.I.} = \frac{P \times 2 \times r}{100} = \frac{2Pr}{100}$$

∴ Difference between C.I. and S.I.

$$\text{C.I.} - \text{S.I.} = \frac{2Pr}{100} + \frac{Pr^2}{100^2} - \frac{2Pr}{100} = \frac{Pr^2}{100^2} = P\left(\frac{r}{100}\right)^2$$

<u>When *n* = 3 years</u>

$$\text{C.I} = P\left(1 + \frac{r}{100}\right)^3 - P$$

$$= P\left[1 + \frac{3r}{100} + \frac{3r^2}{100^2} + \frac{r^3}{100^3}\right] - P$$

$$= P + \frac{3Pr}{100} + \frac{3Pr^2}{100^2} + \frac{Pr^3}{100^3} - P$$

$$\text{C.I.} = \frac{3Pr}{100} + \frac{3Pr^2}{100^2} + \frac{Pr^3}{100^3}$$

$$\text{S.I.} = \frac{P \times 3 \times r}{100} = \frac{3Pr}{100}$$

∴ Difference between C.I. and S.I.

$$\text{C.I.} - \text{S.I.} = \frac{3Pr}{100} + \frac{3Pr^2}{100^2} + \frac{Pr^3}{100^3} - \frac{3Pr}{100} = \frac{3Pr^2}{100^2} + \frac{Pr^3}{100^3}$$

$$= \frac{Pr^2}{100^2}\left(3 + \frac{r}{100}\right)$$

Let us consider an example.

Example 4.1 : Chezhian wants to invest Rs. 10000 in a bank which pays 6% interest per annum for 2 years. He is not able to decide whether to invest in compound interest or simple interest. Kindly advise him.

Compound interest scheme :

Solution : Compound interest = $P\left(1+\frac{r}{100}\right)^n - P$.

P = Rs. 10000, n = 2 years r = 6%.

$$\therefore \text{Compound interest} = 10000\left(1+\frac{6}{100}\right)^2 - 10000$$

$$= 11236 - 10000 = \text{Rs. } 1236$$

Simple Interest Scheme

$$\text{Simple Interest} = \frac{Pnr}{100} = \frac{10000 \times 2 \times 6}{100} = \text{Rs. } 1200$$

$\therefore$ Difference between C.I. and S.I. = 1236 – 1200 = Rs. 36

It is better if he invests in **compound interest scheme.**

Another method :

There is no variation in the principal, time and rate per cent.

Here P = 10000, n = 2 years, r = 6%

Difference between C.I. and S.I. for two years.

$$= \frac{Pr^2}{100^2}$$

$$= \frac{10000 \times 6 \times 6}{100 \times 100} = \text{Rs. } 36$$

$\therefore$ It is better for him to invest in the C.I. scheme.

Example 4.2 : Find the difference between C.I and S.I. on Rs. 8000 at 5% per annum for 3 years.

Solution : Here P = Rs. 8000, n = 3 years, r = 5%

Difference between C.I. and S.I. for 3 years = $\frac{Pr^2}{100^2}\left(3+\frac{r}{100}\right)$

$$\text{C.I.} - \text{S.I.} = 8000 \times \frac{5 \times 5}{100 \times 100}\left(3+\frac{5}{100}\right)$$

$$= 8000 \times \frac{1}{20} \times \frac{1}{20}\left(3+\frac{1}{20}\right)$$

$$= 8000 \times \frac{1}{20} \times \frac{1}{20} \times \frac{61}{20} = \text{Rs. } 61$$

Example 4.3 : Rahul deposited Rs. 5000 in a bank which pays 6% S.I. per annum for 2 years. Ajay deposited on the same day Rs. 5000 in another bank which pays 5.5% C.I. per annum. Who will get more interest and how much ?

Note : Here, we cannot use the formula since the rate of interest differs.

Solution : The S.I. that Rahul will get = $\frac{5000 \times 2 \times 6}{100} = 600$

$$\text{The C.I. that Ajay will get} = 5000\left(1+\frac{5.5}{100}\right)^2 - 5000$$

$$= 5000 \times \frac{105.5}{100} \times \frac{105.5}{100} - 5000$$

$$= 5565.125 - 5000$$

$$= 565.125$$

$$= 565.13$$

Rahul will get 600 – 565.13 = Rs. 34.87 more

Example 4.4 : Abdul borrowed Rs. 6000 from John at 8% per annum. If after 1 year, a dispute arose whether S.I. or C.I. payable half-year should be charged, find the sum in dispute.

Simple Interest :

Solution : P = Rs. 6000 n = 1 year r = 8%

S.I. = $\frac{6000 \times 1 \times 8}{100}$ = Rs. 480

Compound Interest :

P = Rs. 6000 n = 2 half years $r = \frac{8}{2} = 4\%$

$$\text{C.I.} = 6000\left(1+\frac{4}{100}\right)^2 - 6000$$

$$= 6000 \times \frac{104}{100} \times \frac{104}{100} - 6000$$

$$= 6489.60 - 6000$$

$$= 489.60$$

The amount in dispute = 489.60 – 480 = Rs. 9.60

Example 4.5 : Find the difference between C.I. and S.I. on a sum of Rs. 5000 for 2 years at 6% per annum.

Solution : Here P = Rs. 5000, n = 2 years, r = 6%

$$\text{S.I.} = \frac{\text{P}nr}{100} = \frac{5{,}000 \times 2 \times 6}{100} = \text{Rs. } 600$$

$$\text{C.I.} = \text{P}\left[\left(1+\frac{r}{100}\right)^n - 1\right]$$

$$= 5000\left[\left(1+\frac{6}{100}\right)^2 - 1\right]$$

$$= 5000\left[\left(\frac{53}{50}\right)^2 - 1\right]$$

$$= 5000 \times \frac{309}{2{,}500} = \text{Rs. } 618$$

Difference between C.I. and S.I. = 618 – 600 = Rs. 18

Another method :

P = Rs. 5000, r = 6%, n = 2 years

Difference between C.I. and S.I. for 2 years = $P\left(\frac{r}{100}\right)^2 = 5000\left(\frac{6}{100}\right)^2$

$= \text{Rs. } 18$

Difference between C.I. and S.I. = Rs. 18

Example 4.6 : Find the difference between the Compound Interest and Simple Interest on Rs. 8000 at 12½% per annum for 3 years.

Solution : Here P = Rs. 8000, r = 12½%, n = 3 years

$$\text{S.I.} = \frac{Pnr}{100} = \frac{8000 \times 25 \times 3}{2 \times 100} = \text{Rs. } 3000$$

$$\text{Amount (A)} = P\left(1 + \frac{r}{100}\right)^n = 8000\left(1 + \frac{25}{2 \times 100}\right)^3 = 8000\left(1 + \frac{1}{8}\right)^3$$

$$= 8000 \times \frac{9}{8} \times \frac{9}{8} \times \frac{9}{8} = \text{Rs. } 11390.62$$

C.I. = A – P

= 11390.62 – 8000

= Rs. 3390.62

Difference between C.I. and S.I. = 3390.62 – 3000 = Rs. 390.62

Example 4.7 : The difference between S.I. and C.I. for 2 years on a sum of money lent at $6\frac{2}{3}\%$ is Rs. 14. Find the sum.

Solution : Let the principal be Rs. P

Rate (r) = $6\frac{2}{3}\%$

Time (n) = 2 years

Difference between C.I and S.I. for 2 years = Rs. 14

$$P\left(\frac{r}{100}\right)^2 = 14$$

$$P\left(\frac{20}{3 \times 100}\right)^2 = 14$$

$$P\left(\frac{1}{15}\right)^2 = 14$$

$$P \times \frac{1}{15} \times \frac{1}{15} = 14$$

$$P = 14 \times 15 \times 15$$

$$\therefore \text{ Principal} = \text{Rs. } 3150$$

Example 4.8 : Find the Principal if the difference between S.I. and C.I. is Rs. 61 at 5% p.a. in 3 years.

Solution : Let the Principal be Rs. P

$$\text{Rate } (r) = 5\%$$

$$\text{Time } (n) = 3 \text{ years}$$

$$\text{S.I.} = \frac{Pnr}{100} = \frac{P \times 5 \times 3}{100} = \frac{3P}{20}$$

$$\text{C.I.} = P\left[\left(1+\frac{r}{100}\right)^n - 1\right]$$

$$= P\left[\left(1+\frac{5}{100}\right)^3 - 1\right]$$

$$= P\left[\left(1+\frac{1}{20}\right)^3 - 1\right]$$

$$= P\left[\left(\frac{21}{20}\right)^3 - 1\right] = P\left[\frac{9261}{8000} - 1\right]$$

$$\text{C.I.} = \frac{1261}{8000} P$$

$$\text{C.I.} - \text{S.I} = \text{Rs. } 61$$

$$\Rightarrow \quad \frac{1261P}{8000} - \frac{3P}{20} = 61$$

$$\Rightarrow \quad \left[\frac{1261P - 1200P}{8000}\right] = 61$$

$$\Rightarrow \quad \frac{61\,P}{8000} = 61$$

$$\Rightarrow \quad P = 61 \times \frac{8000}{61} = 8000$$

$$\therefore \text{ Principal } = \text{Rs. } 8000$$

Another Method :

The difference between C.I. and S.I. for 3 years = $\frac{P r^2}{100^2}\left(3 + \frac{r}{100}\right)$

$$P \times \frac{5 \times 5}{100 \times 100}\left(3 + \frac{5}{100}\right) = 61$$

$$P \times \frac{1}{20} \times \frac{1}{20}\left(3 + \frac{1}{20}\right) = 61$$

$$P \times \frac{1}{20} \times \frac{1}{20} \times \frac{61}{20} = 61$$

$$P = \frac{61 \times 20 \times 20 \times 20}{61}$$

$$\therefore \text{ Principal } = \text{Rs. } 8000$$

EXERCISE 4.1

1. Find the difference between C.I. and S.I. on a sum of Rs. 15000 for 2 years at 8% p.a.
2. Find the difference between C.I. and S.I. on Rs. 6000 in 2 years at 4% p.a.
3. Find the exact difference between the S.I. and C.I. on Rs. 4500 for 3 years at 10% p.a.
4. Find the difference between C.I. and S.I. on Rs. 32000 at 12% per annum for 3 years
5. The difference between S.I. and C.I. on sum of money lent at 8% p.a. for 2 years is Rs. 12. Find the sum lent.
6. A person borrows a certain sum from a bank at $3\frac{3}{4}$ % S.I. and lends it out for C.I. at the same rate and for the same period of 2 years. If he gains Rs. 27, find the sum borrowed.

7. The difference between the S.I. and C.I. on a certain sum of money at $6\frac{2}{3}$ % for 3 years is Rs. 184. Find the sum.

8. Joesph borrowed Rs. 62500 from Murugan at the rate of 8% per annum. If after $1\frac{1}{2}$ years, a dispute arose whether S.I. or C.I. payable half-year should be charged, find the amount in dispute.

9. Radha borrowed Rs. 6000 at 6% S.I. per annum and immediately lent it to Nargis at 6%. C.I. per annum. How much did she gain at the end of 3 years ?

10. Umayal deposited Rs. 8000 in a bank which pays 8%. S.I. for 2 years. Noorjahan deposited Rs. 8000 in another bank on the same day for 2 years which pays 7.5% C.I. per annum. Who will get more and how much ?

4.2 Recurring Deposit [R.D]

This is a special type of deposit in which a person deposits a fixed sum every month over a period of years and receives a large sum at the end of the specified number of years. It enables salaried people to save a small sum every month and get a large sum after a few years. Since the deposit is made month after month it is called recurring deposit. Recurring deposits are also known as Cumulative Term Deposits. The amount deposited every month is called the monthly deposit.

For recurring deposits carrying interest, the maturity value is calculated as follows :

Let P be the monthly instalment and R% be the rate of interest and '*n*' be the number of monthly instalments, then the first instalment is retained by the bank for '*n*' months, the second for $(n-1)$ months and so on and the last instalment for 1 month.

$$\therefore \text{ Total interest } = P \times \frac{n}{12} \times \frac{R}{100} + P \times \frac{(n-1)}{12} \times \frac{R}{100} + \dots + P \times \frac{2}{12} \times \frac{R}{100} + P \times \frac{1}{12} \times \frac{R}{100}$$

$$= P \times \frac{R}{100} \times \frac{1}{12}[n + (n-1) + (n-2) + \dots + 2 + 1]$$

$$= \frac{PR}{100} \times \frac{1}{12} \times \frac{n(n+1)}{2}$$

$$= \frac{PNR}{100} \text{ where } N = \frac{n(n+1)}{2 \times 12} \text{ years}$$

$$\text{Interest} = \frac{PNR}{100}$$

$$\text{Amount due} = \text{Amount deposited} + \text{Interest}$$

$$= Pn + \frac{PNR}{100}$$

Example 4.9 : Arun deposits Rs. 300 per month for 2 years in a bank which pays 10% S.I. per annum on R.D. Find the amount he gets at the end of 24 months.

Solution : Monthly deposit (P) = Rs. 300

Rate of interest (R) = 10%

Time (n) = 2 years

$= 2 \times 12 = 24$ months

$$N = \frac{n(n+1)}{2 \times 12} = \frac{24 \times 25}{2 \times 12} = 25 \text{ years}$$

$$\text{Amount (A)} = Pn + \frac{PNR}{100}$$

$$= (300 \times 24) + \frac{300 \times 25 \times 10}{100}$$

$$= 7200 + 750 = 7950$$

Amount = Rs. 7950

Example 4.10 : A person opens an R.D. account paying Rs. 150 per month for 3 years. If the rate of interest is 12%, what is the amount of interest he gets at the end ?

Solution : Monthly deposit (P) = Rs. 150

Rate of Interest (R) = 12%

Time (n) = 3 years

$= 3 \times 12$

n = 36 months

$$N = \frac{n(n+1)}{2 \times 12}$$

$$= \frac{36 \times 37}{2 \times 12} = \frac{111}{2} \text{ years}$$

$$\text{Interest} = \frac{PNR}{100}$$

$$= \frac{150 \times 111 \times 12}{2 \times 100}$$

$$= \text{Rs. } 999$$

$$\text{Amount of Interest} = \text{Rs. } 999$$

Example 4.11 : How much one should deposit every month in a bank paying 5% S.I. per annum on monthly R.D, if at the end of 6 years one wants to get Rs. 3318 ?

Solution : Let the monthly deposit be Rs. P

$$\text{Rate of Interest (R)} = 5\%$$

$$\text{Time } (n) = 6 \text{ years} = 6 \times 12$$

$$n = 72 \text{ months}$$

$$N = \frac{n(n+1)}{2 \times 12}$$

$$= \frac{72 \times 73}{2 \times 12} = 219 \text{ years}$$

$$\text{Amount (A)} = Pn + \frac{PNR}{100}$$

$$(P \times 72) + \left(\frac{P \times 219 \times 5}{100}\right) = 3318$$

$$72P + \frac{219P}{20} = 3318$$

$$\frac{1440P + 219P}{20} = 3318$$

$$\frac{1659P}{20} = 3318$$

$$P = \frac{3318 \times 20}{1659} = 40$$

$$\text{Monthly deposit} = \text{Rs. } 40$$

Example 4.12 : Ramya invested Rs. 500 every month for 2 years in a bank and collects Rs. 12500 at the end of 2 years. Find the rate of simple interest paid by the bank on recurring deposit ?

Solution : Monthly deposit (P) = Rs. 500

Let the rate of interest be R%

$$\text{Time } (n) = 2 \text{ years}$$
$$= 2 \times 12$$
$$n = 24 \text{ months}$$
$$N = \frac{n(n+1)}{2 \times 12}$$
$$= \frac{24 \times 25}{2 \times 12}$$
$$= 25 \text{ years}$$
$$Pn + \frac{PNR}{100} = A$$
$$(500 \times 24) + \frac{500 \times 25 \times R}{100} = 12500$$
$$12000 + 125\,R = 12500$$
$$125\,R = 12500 - 12000$$
$$125\,R = 500$$
$$R = \frac{500}{125}$$
$$= 4\%$$
$$\therefore \text{ Rate of interest} = 4\%$$

EXERCISE 4.2

1. Balu deposits Rs. 250 per month for 2 years in a bank which pays 12% simple interest on recurring deposits. Find the amount he gets at maturity.
2. Ebi deposits Rs. 100 every month in a bank paying 8% p.a. S.I. on R.D. How much will she get at the end of 60 months ?

3. A person deposits Rs. 40 in a bank every month at 10% S.I. How much will he get at the end of 3 years ?

4. At the end of 3 years a recurring deposit fetches. Rs. 16398 with 9% simple interest per annum. Find the amount to be deposited every month.

5. A bank pays 8% simple interest per annum on recurring deposits. If Selva wants to get an amount of Rs. 8088 at the end of 3 years, find the monthly instalment.

6. How much should Sneka deposit at the beginning of every month in a bank paying 5% S.I, if she wants to get Rs. 6636 at the end of 6 years ?

7. A person remits a sum of Rs. 500 per month into a R.D. account of a bank. He gets a sum of Rs. 21663 at the end of 3 years. What is the rate of interest given by the bank ?

8. Sita invests Rs. 25 in a bank at the beginning of each month for 36 months. If she gets Rs. 1066.50 at the end of 36 months, find the rate of interest.

9. Aravindh deposited Rs. 200 at the beginning of every month in recurring deposit and received Rs. 19656 at the end of 6 years. Find the rate of simple interest paid by the bank.

4.3 FIXED DEPOSIT

Fixed deposits are deposits for a fixed period of time and the depositor can withdraw his money only after the expiry of the fixed period. It is also known as Term Deposits. However, in the case of necessity, the depositor can get his fixed deposit terminated earlier or get a loan from the bank under terms laid down by the bank. There are two types of fixed deposits, namely (i) Short term deposits and (ii) Long term deposits.

Short term Deposits : Fixed deposits are accepted by the banks for a short period ranging from 46 days to one year. The interest paid on this deposit is simple interest.

Long term Deposits : Fixed deposits are accepted by the banks for a period of one year or more. The interest paid on this type of deposit is compound interest.

Formulae : Quarterly Interest = $\frac{Pr}{400}$

Half yearly interest = $\frac{Pr}{200}$

Example 4.13 : What is the half yearly interest received for Rs. 5000 in a bank on a fixed deposit at the rate of interest 10% for 2 years.

Solution : Principal (P) = Rs. 5000

Rate (r) = 10%

Time (n) = $\frac{1}{2}$ year

Interest = $\frac{Pnr}{100}$

$= \frac{5000 \times 1 \times 10}{2 \times 100} = 250$

Half yearly interest = Rs. 250

Example 4.14 : Radha made a fixed deposit with a bank for 3 years paying 11% p.a. If she takes quarterly interest, find the interest she gets on Rs. 1000 deposit ?

Solution : Principal (P) = Rs. 1000

Rate (r) = 11 %

Time (n) = 1/4 year

Quarterly Interest = $\frac{Pnr}{100} = \frac{1000 \times 1 \times 11}{4 \times 100}$ = Rs. 27.50

Quarterly Interest = Rs. 27.50

Example 4.15 : If the quarterly interest on a sum kept in fixed deposit with a bank for 2 years paying 9% p.a. was Rs. 540, find the amount of deposit.

Solution : Let the amount of deposit be Rs. P.

Rate (r) = 9%

Time (n) = 1/4 year

Quarterly interest = Rs. 540

$$\frac{Pnr}{100} = 540$$

$$\frac{P \times 1 \times 9}{4 \times 100} = 540$$

$$9P = 540 \times 400$$

$$P = \frac{540 \times 400}{9} = 24000$$

Amount deposited = Rs. 24000

Example 4.16 : Sheba deposited Rs. 14000 as a special deposit for 3 years and the interest was compounded yearly at the rate of 10% p.a. Find the maturity value of the deposit.

Solution : Principal (P) = Rs. 14000

Rate (r) = 10%

Time (n) = 3 years

$$\text{Amount (A)} = P\left[1+\frac{r}{100}\right]^n = 14000\left[1+\frac{10}{100}\right]^3$$

$$= 14000\left[1+\frac{1}{10}\right]^3 = 14000\left(\frac{11}{10}\right)^3$$

$$= 14000 \times \frac{11}{10} \times \frac{11}{10} \times \frac{11}{10} = 18634$$

Maturity value = Rs. 18634

Example 4.17 : Which is better investment : Rs. 2000 in a fixed deposit with a bank for 3 years, the interest being compounded halfyearly at the rate of 10% **(or)** Rs. 60 per month in a recurring deposit with a bank paying simple interest of 10% per annum for 36 months.

Solution : Fixed Deposit :

Principal (P) = Rs. 2000

Half yearly Rate (r) = $\frac{10}{2}\% = 5\%$

Time = 3 × 2 = 6 half year.

$$A = P\left(1+\frac{r}{100}\right)^n$$

$$\text{Amount} = 2000\left(1+\frac{5}{100}\right)^6$$

$$A = 2000\,(1.05)^6$$

Taking log log A = log 2000 + 6 log 1.05

= 3.3010 + 6 (0.0212)

= 3.3010 + 0.1272 = 3.4282

$$A = \text{Antilog } 3.4282 = 2680$$

Maturity value = Rs. 2680

Recurring Deposit :

P = Rs. 60, R = 10%, n = 36 months

$$N = \frac{n\,(n+1)}{2 \times 12} = \frac{36 \times 37}{2 \times 12} = \frac{111}{2} \text{ years}$$

$$A = Pn + \frac{PNR}{100}$$

$$= (60 \times 36) + \frac{60 \times 111 \times 10}{2 \times 100}$$

$$= 2160 + 333 = \text{Rs. } 2493$$

Amount due = Rs. 2493

∴ The fixed deposit investment is better.

EXERCISE 4.3

1. Swamy deposited Rs. 3000 in a bank as a fixed deposit for 2 years paying 10% p.a. and receives interest halfyearly. Find the interest received by him in 2 years.
2. Rani made a fixed deposit with a bank for 1½ years paying interest 9% p.a. If she gets quarterly interest, find the interest she gets on Rs. 12000 quarterly.
3. Mahalakshmi made a special fixed deposit of Rs. 9000 with a bank for 3 years paying interest at 10% p.a. If the bank compounded interest half yearly, find the maturity value of the deposit.
4. If Karthik had a special deposit of Rs. 12000 with a bank for 3 years at the rate of 10% p.a. and the interest was compounded yearly, how much interest would have been paid to him ?
5. Which is a better investment ? Rs. 5000 in a special deposit with a bank for 3 years the interest being compounded half yearly at the rate of 10% p.a. (or) Rs. 150 per month in a recurring deposit with a bank paying simple interest of 10% p.a. for 36 months.

5. ALGEBRA

5.1 SIMULTANEOUS EQUATIONS

In earlier classes, we have learnt different methods of solving simultaneous linear equations with two variables. Here, we shall see how to solve the linear equations of the form $ax + by + cz + d = 0$ where a, b, c, d are real numbers with $a \neq 0$, $b \neq 0$ and $c \neq 0$.

Rules for solving three linear equations :

1. Eliminate one of the variables from any pair of the equations.
2. Eliminate the same variable from another pair.
3. Two equations involving two variables are thus obtained.
4. Solve them in the usual way studied in earlier classes.
5. The remaining variable is then found by substituting in any one of the given equations.

Example 5.1 : Solve : $6x + 2y - 5z = 13$; $3x + 3y - 2z = 13$; $7x + 5y - 3z = 26$

Solution :

$$6x + 2y - 5z = 13 \quad \text{.... (1)}$$
$$3x + 3y - 2z = 13 \quad \text{.... (2)}$$
$$7x + 5y - 3z = 26 \quad \text{.... (3)}$$

(1) × 3 ⇒ $18x + 6y - 15z = 39$

(2) × 2 ⇒ $6x + 6y - 4z = 26$

Subtracting, $12x - 11z = 13$ .... (4)

(1) × 5 ⇒ $30x + 10y - 25z = 65$

(3) × 2 ⇒ $14x + 10y - 6z = 52$

Subtracting, $16x - 19z = 13$ ... (5)

(4) × 4 ⇒ $48x - 44z = 52$

(5) × 3 ⇒ $48x - 57z =39$

Subtracting, $13z = 13$

⇒ $z = 1$

from (4) and (1) $x = 2$, $y = 3$

$\therefore$ Solution is $x = 2$; $y = 3$; $z = 1$

Example 5.2 : $\frac{x}{2} - 1 = \frac{y}{6} + 1 = \frac{z}{7} + 2$; $\frac{y}{3} + \frac{z}{2} = 13$

Solution : From the equation: $\frac{x}{2} - 1 = \frac{y}{6} + 1$

We have $\quad 3x - y = 12 \quad$ ... (1)

From the equation $\quad \frac{x}{2} - 1 = \frac{z}{7} + 2$

We have $\quad 7x - 2z = 42 \quad$ ... (2)

And from the equation $\quad \frac{y}{3} + \frac{z}{2} = 13,$

we have $\quad 2y + 3z = 78 \quad$ ... (3)

Eliminating z from (2) and (3) we have

$$21x + 4y = 282$$

and from (1) $\quad 12x - 4y = 48$

Hence $\quad x = 10, \quad y = 18$

Also by substitution in (2) , $\; z = 14$

$\therefore$ Solution is (x, y, z) is (10, 18, 14).

Example 5.3 : Solve $\frac{1}{2x} + \frac{1}{4y} - \frac{1}{3z} = \frac{1}{4}$; $\frac{1}{x} = \frac{1}{3y}$; $\frac{1}{x} - \frac{1}{5y} + \frac{4}{z} = 2\frac{2}{15}$

Solution : $\quad \frac{1}{2x} + \frac{1}{4y} - \frac{1}{3z} = \frac{1}{4} \quad$ ... (1)

$$\frac{1}{x} = \frac{1}{3y} \quad \text{... (2)}$$

$$\frac{1}{x} - \frac{1}{5y} + \frac{4}{z} = 2\frac{2}{15} \quad \text{... (3)}$$

Clearing of fractional coefficients, we obtain

From (1) $\quad \frac{6}{x} + \frac{3}{y} - \frac{4}{z} = 3 \quad$ ... (4)

From (2) $\frac{3}{x} - \frac{1}{y} = 0$ ... (5)

From (3) $\frac{15}{x} - \frac{3}{y} + \frac{60}{z} = 32$ ... (6)

Multiply (4) by 15 and add the result to (6) we have

$$\frac{105}{x} + \frac{42}{y} = 77,$$

dividing by 7, $\frac{15}{x} + \frac{6}{y} = 11$ ... (7)

From (5) $\frac{18}{x} - \frac{6}{y} = 0$

$$\therefore \frac{33}{x} = 11$$

$\Rightarrow$ $x = 3$

From (5) $y = 1$

From (4) $z = 2$

EXERCISE - 5.1

Solve the following simultaneous equations.

1. $x + 2y + 2z = 11$, $2x + y + z = 7$, $3x + 4y + z = 14$
2. $3x - 2y + z = 2$, $2x + 3y - z = 5$, $x + y + z = 6$
3. $x + 2y + z = 7$, $x + 3z = 11$, $2x - 3y = 1$
4. $2x - y = 4$, $y - z = 6$, $x - z = 10$
5. $3x - 4y = 6z - 16$, $4x - y - z = 5$, $x = 3y + 2(z - 1)$
6. $x - \frac{y}{5} = 6$, $y - \frac{z}{7} = 8$, $z - \frac{x}{2} = 10$
7. $\frac{y+z}{4} = \frac{z+x}{3} = \frac{x+y}{2}$, $x + y + z = 27$
8. $\frac{1}{x} - \frac{2}{y} + 4 = 0$, $\frac{1}{y} - \frac{1}{z} + 1 = 0$, $\frac{2}{z} + \frac{3}{x} = 14$

9. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=36,\ \frac{1}{x}+\frac{3}{y}-\frac{1}{z}=28,\ \frac{1}{x}+\frac{1}{3y}+\frac{1}{2z}=20.$

10. $\frac{9}{x}-\frac{2}{y}=\frac{5}{z}-\frac{3}{x}=\frac{7}{y}+\frac{15}{2z}=4$

11. $\frac{1}{3}(x+y-5)=y-z=2x-11=9-(x+2z)$

12. $x+20=\frac{3y}{2}+10$
$=2z+5$
$=110-(y+z)$

Applications to word problems

In this section, we shall learn about some applications of simultaneous linear equations in solving problems related to our day - to - day life. In solving such problems, we may use the following rules :

1. Read the problem carefully and identify the unknown quantities.
2. Give to each of these quantities a variable name like x, y, z or a, b, c, etc.
3. Identify the variables to be determined.
4. Formulate the equations in terms of the variables to be determined.
5. Solve the equations using any one of the methods learnt earlier.

Example 5.4 : In a ration shop the sale of sugar, rice and wheat were as follows:

	Sugar (in kg)	Rice (in kg)	Wheat (in kg)	Sale (Amount)
Monday	1	4	3	78
Tuesday	2	5	7	126
Wednesday	1	6	4	108

Find the sale price of each item per kg.

Solution : Let the sale price of 1 kg of sugar, rice and wheat be Rs.x, Rs. y and Rs.z respectively. Then we get,

$$x+4y+3z=78 \quad \text{... (1)}$$
$$2x+5y+7z=126 \quad \text{... (2)}$$
$$x+6y+4z=108 \quad \text{... (3)}$$

$$
\begin{array}{llrl}
(1) \times 2 & \Rightarrow & 2x + 8y + 6z & = 156 \\
(2) & \Rightarrow & 2x + 5y + 7z & = 126 \\
\hline
\text{Subtracting} & & 3y - z & = 30 \quad \ldots (4) \\
\hline
(3) - (1) & \Rightarrow & 2y + z & = 30 \quad \ldots (5) \\
(4) + (5) & \Rightarrow & 5y & = 60 \\
 & & \therefore y & = 12 \\
\text{from } (5) & & z & = 6 \\
\text{from } (1) & & x & = 12
\end{array}
$$

$\therefore$ Sale price per kg of sugar, rice and wheat is Rs. 12, Rs. 12 and Rs. 6 respectively.

EXERCISE 5.2

1. Vinu bought 5 rulers, 7 ink pads and 3 pens at a cost of Rs. 52. Rosy bought 4 pens, 6 ink pads and 7 rulers for Rs.53 when Paul bought 7 pens and 3 ink pads, the shop keeper took a 50 rupee note from him and paid back the cost of 7 rulers.Find the cost of each.
2. In a shop, items A, B, C were sold as follows to three different persons.

   3A's, 2B's and 5C's; 5A's, 1 B's and 5C's ; 3A's, 5B's and 6C's. If the persons paid Rs. 22, Rs.23 and Rs. 33, find the sale price of each item.
3. There are 3 types of benches in a class. If a class of 100 students is seated using 5 benches of type A, 4 benches of type B and 3 benches of type C, 6 students go without seats. If 4 benches of type A , 5 benches of type B and 3 benches of type C are used then 2 students are left out with no seats. When 3 benches of type A , 7 benches of type B and 2 benches of type C are used then there are seats for 4 more students. Find the no. of students seated on each bench.
4. A purse contains five, ten and twenty rupees currency. There are 12 pieces. Its total value is Rs. 105. But when first 2 sorts are interchanged in their numbers,its value will be increased by Rs. 20. Find the numbers of currencies in each sort.
5. Sum of 3 numbers is 10. Sum of the first number, twice the second number and 3 times the third is 19 and the sum of first, four times the second and nine times the third is 43, Find the numbers.

5.2 REMAINDER THEOREM, FACTOR THEOREM AND SYNTHETIC DIVISION

5.2.1 Remainder Theorem

If a polynomial $p(x)$ of degree greater than or equal to one, over the set of real numbers (R) is divided by $x - a$, where $a \in$ R, then the remainder is $p(a)$.

If $(x + a)$ divides $p(x)$ then the remainder becomes $p(-a)$. Similarly if $(ax + b)$ divides $p(x)$, the remainder is $p\left(\frac{-b}{a}\right)$ and if $(ax - b)$ divides $p(x)$ the remainder is $p\left(\frac{b}{a}\right)$.

Example 5.5 : Find the remainder when the polynomial $p(x)=x^4 - 3x^2+2x+ 1$ is divided by $x - 1$.

Solution : By Remainder theorem, the required remainder is equal to $p(1)$ [(i.e.,) as $x - 1$ is the divisor, $p(1)$ is the remainder]

$$\text{Now } p(x) = x^4 - 3x^2 + 2x + 1$$

$$\text{Remainder } p(1) = (1)^4 - 3(1)^2 + 2(1) + 1$$

$$= 1 - 3 + 2 + 1$$

$$p(1) = 1$$

Example 5.6 : Find the remainder when $2x^4 - 4x^3 + x^2 - 8x +3$ is divided by $2x + 1$.

Solution : As $2x + 1$ is the divisor, $p\left(\frac{-1}{2}\right)$ is the remainder.

$$\text{Let } p(x) = 2x^4 - 4x^3 + x^2 - 8x + 3$$

$$p\left(\frac{-1}{2}\right) = 2\left(-\frac{1}{2}\right)^4 - 4\left(-\frac{1}{2}\right)^3 + \left(-\frac{1}{2}\right)^2 - 8\left(-\frac{1}{2}\right) + 3$$

$$= \frac{1}{8} + \frac{1}{2} + \frac{1}{4} + 4 + 3 = \frac{1+4+2}{8} + 7 = 7\frac{7}{8}.$$

Example 5.7 : Find the remainder when $f(x) = x^3 - 6x^2 + 2x - 4$ is divided by $1 - 3x$.

Solution : By remainder theorem, the remainder is $f\left(\frac{1}{3}\right)$.

$$f(x) = x^3 - 6x^2 + 2x - 4$$

$$f\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^3 - 6\left(\frac{1}{3}\right)^2 + 2\left(\frac{1}{3}\right) - 4$$

$$= \frac{1}{27} - \frac{6}{9} + \frac{2}{3} - 4 = \frac{1-18+18-108}{27} = \frac{-107}{27}$$

Example 5.8 : If the polynomials $ax^3 + 4x^2 + 3x - 4$ and $x^3 - 4x + a$ leave the same remainder when divided by $(x - 3)$, find the value of a.

Solution : Let $p(x) = ax^3 + 4x^2 + 3x - 4$ and $q(x) = x^3 - 4x + a$

The remainders when $p(x)$ and $q(x)$ are divided by $(x - 3)$ are $p(3)$ and $q(3)$ respectively.

Given that $p(3) = q(3)$.

$\Rightarrow a\,(3)^3 + 4\,(3)^2 + 3\,(3) - 4 = 3^3 - 4\,(3) + a$

$\Rightarrow \quad 27a + 36 + 9 - 4 = 27 - 12 + a$

$\Rightarrow \quad 26a + 26 = 0 \Rightarrow 26a = -26$

$\Rightarrow \quad a = \frac{-26}{26} \Rightarrow \quad a = -1$

Example 5.9 : R_1 and R_2 are the remainders when the polynomials $x^3 + 2x^2 - 5ax - 7$ and $x^3 + ax^2 - 12x + 6$ are divided by $x + 1$ and $x - 2$ respectively. If $2R_1 + R_2 = 6$, find the value of a.

Solution : Let $p(x) = x^3 + 2x^2 - 5ax - 7$ and

$q(x) = x^3 + ax^2 - 12x + 6$

Now $R_1 = p(-1)$

$= (-1)^3 + 2\,(-1)^2 - 5a\,(-1) - 7$

$= -1 + 2 + 5a - 7$

$R_1 = 5a - 6$

and $R_2 = q(2)$

$= (2)^3 + a(2)^2 - 12(2) + 6$

$= 8 + 4a - 24 + 6$

$R_2 = 4a - 10$

given $2R_1 + R_2 = 6$

$\therefore\ 2(5a - 6) + (4a - 10) = 6$

$\Rightarrow \quad 14a = 28 \Rightarrow \quad a = 2$

Example 5.10 : If the remainder, when $a - 2x + 5x^2$ divided by $x - 2$ is 7, then find a.

Solution : Let $p(x) = a - 2x + 5x^2$

given that $p(2) = 7$

$\Rightarrow \quad a - 2(2) + 5(2)^2 = 7$

$\Rightarrow \quad a - 4 + 20 = 7$

$\Rightarrow \quad a = -9$

Example 5.11 : Given that $px^2 + qx + 6 = 0$ leaves a remainder 1 on division by $2x + 1$ and $2qx^2 + 6x + p$ leaves a remainder 2 on division by $3x - 1$. Find p and q.

Solution : Let $f(x) = px^2 + qx + 6$ and $q(x) = 2qx^2 + 6x + p$

By data, $f\left(\frac{-1}{2}\right) = 1$

$$\Rightarrow p\left(\frac{-1}{2}\right)^2 + q\left(\frac{-1}{2}\right) + 6 = 1$$

$$\frac{p}{4} - \frac{2q}{4} = -5$$

$$p - 2q = -20 \quad \text{... (1)}$$

Also, $q\left(\frac{1}{3}\right) = 2$

$$2q\left(\frac{1}{3}\right)^2 + 6\left(\frac{1}{3}\right) + p = 2$$

$$\frac{2q}{9} + p = 0$$

$$2q + 9p = 0 \quad \text{... (2)}$$

$(1) \Rightarrow \quad p - 2q = -20$

$(2) \Rightarrow \quad 9p + 2q = 0$

$$10p = -20$$

$$\therefore\ p = -2$$

from (2), we get $q = 9$

EXERCISE - 5.3

1. Find the remainder on dividing $f(x)$ by $(x + 3)$ where (i) $f(x) = 2x^2 - 7x - 1$ (ii) $f(x) = 3x^3 - 7x^2 + 11x + 1$.
2. Find the remainder when $4x^3 - 3x^2 + 2x - 4$ is divided by (i) $x - 2$, (ii) $x + \frac{1}{2}$.
3. Find m if $5x^7 - 9x^3 + 3x - m$ leaves a remainder 7 when divided by $x + 1$.
4. When $x^3 + 3x^2 - kx + 4$ is divided by $x - 2$, the remainder is k. Find the value of k.
5. Find the value of p if the division of $px^3 + 9x^2 + 4x - 10$ by $x + 3$ leaves the remainder 5.
6. If the polynomials $ax^3 + 3x^2 - 13$ and $2x^3 - 5x + a$ leave the same remainder when divided by $x + 2$, find the value of a.
7. If the polynomials $2x^3 + ax^2 + 3x - 5$ and $x^3 + x^2 - 4x + a$ leave the same remainder when divided by $x - 2$, find the value of a.
8. The polynomials $ax^3 + 3x^2 - 3$ and $2x^3 - 5x + a$ when divided by $x - 4$ leave the remainders R_1 and R_2 respectively. Find the values of a in each of the following cases, if

   (i) $R_1 = R_2$ (ii) $R_1 + R_2 = 0$ (iii) $2R_1 - R_2 = 0$.
9. If $x^3 + lx + m$ leaves the same remainder 7, when divided by $x - 1$ or by $x + 1$, find l and m.
10. The expression $2x^3 + ax^2 + bx - 2$ leaves the remainder 7 and 0 when divided by $(2x - 3)$ and $(x + 2)$ respectively. Calculate the values of a and b.
11. The remainders when $px^3 - qx^2 - x + 5$ and $x^3 + px^2 + 2x - q - 4$ are divided by $x + 1$ are -3 and 10 respectively. Find p and q.
12. Given that $px^3 + 9x^2 + qx + 1$ leaves remainder 4 on division by $2x + 1$ and $9x^3 + qx^2 + px + 1$ leaves the remainder 3 on division by $3x - 1$, find p and q.

5.2.2 Factor Theorem

If $p(x)$ is a polymial of degree $n \geq 1$ and a is any real number, then

(i) $(x - a)$ is a factor of $p(x)$ if $p(a) = 0$ and

(ii) $p(a) = 0$ if $(x - a)$ is a factor of $p(x)$.

Example 5.12 : Show that $(x - 3)$ is a factor of the polynomial $x^3 - 3x^2 + 4x - 12$.

Solution : Let $p(x) = x^3 - 3x^2 + 4x - 12$

Here, $p(3) = 3^3 - 3(3)^2 + 4(3) - 12$

$= 27 - 27 + 12 - 12 = 0$

Hence $(x - 3)$ is a factor of $p(x) = x^3 - 3x^2 + 4x - 12$.

Example 5.13 : Show that $(2x - 3)$ is a factor of $2x^3 - 9x^2 + x + 12$

Solution : Let $p(x) = 2x^3 - 9x^2 + x + 12$

$$\text{Here, } p\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - 9\left(\frac{3}{2}\right)^2 + \frac{3}{2} + 12$$

$$= \frac{27 - 81 + 6 + 48}{4} = 0$$

Hence $(2x - 3)$ is a factor of $2x^3 - 9x^2 + x + 12$.

Example 5.14 : Show that $(x + 2)$ and $(3x + 1)$ are both factors of $6x^3 + 11x^2 - 3x - 2$.

Solution : Let $f(x) = 6x^3 + 11x^2 - 3x - 2$

$f(-2) = 6(-2)^3 + 11(-2)^2 - 3(-2) - 2$

$= -48 + 44 + 6 - 2 = 0$

$\therefore$ $(x + 2)$ is a factor of $f(x)$

$$f\left(\frac{-1}{3}\right) = 6\left(\frac{-1}{3}\right)^3 + 11\left(\frac{-1}{3}\right)^2 - 3\left(\frac{-1}{3}\right) - 2$$

$$= 6\left(\frac{-1}{27}\right) + 11\left(\frac{1}{9}\right) + 1 - 2$$

$$= \frac{-2}{9} + \frac{11}{9} + 1 - 2 = 0$$

$\therefore$ $(3x + 1)$ is a factor of $f(x)$

Example 5.15 : If $f(x) = 2x^4 - 17x^3 + 49x^2 - 52x + 12$ and $g(x) = x^2 - 5x + 6$, show that $g(x)$ is a factor of $f(x)$.

Solution : $g(x) = x^2 - 5x + 6$

$= (x - 2)(x - 3)$ [by factorising]

$$f(x) = 2x^4 - 17x^3 + 49x^2 - 52x + 12$$

$$f(2) = 2(2)^4 - 17(2)^3 + 49(2)^2 - 52(2) + 12$$

$$= 32 - 136 + 196 - 104 + 12 = 0$$

$\therefore$ $(x - 2)$ is a factor of $f(x)$ ... (1)

Again, $f(3) = 2(3)^4 - 17(3)^3 + 49(3)^2 - 52(3) + 12$

$$= 162 - 459 + 441 - 156 + 12 = 0$$

$\therefore$ $(x - 3)$ is a factor of $f(x)$ ... (2)

from (1) and (2),

$$(x - 2)(x - 3) = x^2 - 5x + 6 = g(x) \text{ is a factor of } f(x)$$

Example 5.16 : For what value of m is $2x^3 - x^2 - 3mx - 24$ exactly divisible by $x - 2$?

Solution : Let $f(x) = 2x^3 - x^2 - 3mx - 24$

Since $(x - 2)$ is a factor of $f(x)$, by factor theorem,

$$f(2) = 0$$

$\Rightarrow 2(2)^3 - 2^2 - 3m(2) - 24 = 0$

$\Rightarrow \quad 16 - 4 - 6m - 24 = 0$

$\Rightarrow \quad m = -2$

Example 5.17 : Find the value of k if $(x - 1)$ exactly divides $k^2x^2 + 3kx - (3k + 4)$

Solution : Let $p(x) = k^2x^2 + 3kx - (3k + 4)$

Since $(x - 1)$ is a factor, by factor theorem,

$$P(1) = 0$$

(i.e.,) $k^2(1)^2 + 3k(1) - (3k + 4) = 0$

$$k^2 = 4 \Rightarrow k = \pm 2.$$

Example 5.18 : Find the values of p and q so that $(x + 2)$ and $(x - 1)$ are factors of the polynomial $x^3 + 10x^2 + px + q$.

Solution : Let $f(x) = x^3 + 10x^2 + px + q$

Since $(x + 2)$ is a factor of $f(x)$, $f(-2) = 0$

$\Rightarrow \quad (-2)^3 + 10(-2)^2 + p(-2) + q = 0$

$\Rightarrow \quad -8 + 40 - 2p + q = 0$

$-2p + q = -32 \quad \ldots (1)$

Since $(x - 1)$ is a factor of $f(x)$, $f(1) = 0$

$\Rightarrow \quad 1^3 + 10(1)^2 + p(1) + q = 0$

$\Rightarrow \quad 1 + 10 + p + q = 0$

$\Rightarrow \quad p + q = -11 \quad \ldots (2)$

$(1) - (2) \Rightarrow \quad -3p = -21$

$$\Rightarrow \quad p = \frac{-21}{-3} = 7$$

from (2) $\quad 7 + q = -11$

$q = -11 - 7 = -18$

$\therefore\ p = 7\ ;\ q = -18$

Example 5.19 : If $ax^3 + bx^2 + x - 6$ has $x + 2$ as a factor and leaves a remainder 4 when divided by $(x - 2)$, find the values of a and b.

Solution : Let $p(x) = ax^3 + bx^2 + x - 6$

Since $(x + 2)$ is a factor, $p(-2) = 0$

$\Rightarrow \quad a(-2)^3 + b(-2)^2 + (-2) - 6 = 0$

$\Rightarrow \quad -8a + 4b = 8$

$\Rightarrow \quad -2a + b = 2 \quad \ldots (1)$

Also given that $p(x)$ leaves the remainder 4 when divided by $x - 2$

$\therefore\ p(2) = 4$

$\Rightarrow \quad a(2)^3 + b(2)^2 + 2 - 6 = 4$

$\Rightarrow \quad 8a + 4b = 8$

$\Rightarrow \quad 2a + b = 2 \quad \ldots (2)$

$(1) + (2) \Rightarrow \quad 2b = 4$

$$b = \frac{4}{2} = 2$$

Substituting $b = 2$ in (2), we get $a = 0$

$$\therefore\ a = 0;\ b = 2$$

Example 5.20 : If both $x - 2$ and $x - \frac{1}{2}$ are factors of $px^2 + 5x + r$, show that $p = r$.

Solution : Let $f(x) = px^2 + 5x + r$

Since $(x - 2)$ is a factor of $f(x)$, $f(2) = 0$

$$\Rightarrow\quad p(2)^2 + 5(2) + r = 0$$

$$\Rightarrow\quad 4p + 10 + r = 0$$

$$\Rightarrow\quad 4p + r = -10 \qquad \text{... (1)}$$

Since $(x - \frac{1}{2})$ is a factor of $f(x)$, $f\left(\frac{1}{2}\right) = 0$

$$\Rightarrow\quad p\left(\frac{1}{2}\right)^2 + 5\left(\frac{1}{2}\right) + r = 0$$

$$\Rightarrow\quad \frac{p}{4} + \frac{5}{2} + r = 0$$

$$\Rightarrow\quad \frac{p+4r}{4} = \frac{-5}{2}$$

$$\Rightarrow\quad p + 4r = -10 \qquad \text{... (2)}$$

$\therefore$ from (1) and (2), we get $4p + r = p + 4r$

$$\Rightarrow\quad 3p = 3r$$

$$\Rightarrow\quad p = r$$

Example 5.21 : If $x^2 - 1$ is a factor of $ax^4 + bx^3 + cx^2 + dx + e$, show that $a + c + e = b + d = 0$

Solution : Let $p(x) = ax^4 + bx^3 + cx^2 + dx + e$

Given $x^2 - 1$ is a factor of $p(x)$

$\Rightarrow (x - 1)$ and $(x + 1)$ are factors of $p(x)$

$\Rightarrow p(1) = 0$ and $p(-1) = 0$

$\Rightarrow a + b + c + d + e = 0$ and $a - b + c - d + e = 0$

Adding and subtracting these two equations we get,

$2\,(a + c + e) = 0$ and $2\,(b + d) = 0$

$$\Rightarrow (a + c + e) = 0 \text{ and } (b + d) = 0$$

$$\Rightarrow a + c + e = b + d = 0$$

EXERCISE 5.4

1. Show that $(x - 5)$ and $(2x - 1)$ are factors of $2x^2 - 11x + 5$.
2. Show that $2x + 7$ is a factor of $2x^3 + 7x^2 - 4x - 14$.
3. Find the value of k, if $(x + 3)$ is a factor of $3x^2 + kx + 6$.
4. Find the value of a, if $(x - a)$ is a factor of $x^3 - ax^2 + x + 2$
5. Find the value of k, if $(x + 2)$ is a factor of $(x + 1)^7 + 2x + k$.
6. Determine the value of a for which the polynomial $2x^4 - ax^3 + 4x^2 + 2x + 1$ is divisible by $1 - 2x$.
7. Find the value of a, if $x + a$ is a factor of $x^3 + ax^2 - 2x + a + 4$.
8. Using factor theorem, show that $x + a$ is a factor of $x^n + a^n$ when n is any odd positive integer.
9. If $f(x) = x^4 - 5x^2 + 4$ and $g(x) = x^2 - 3x + 2$, show that $g(x)$ is a factor of $f(x)$.
10. If $(x - 1)$ and $(x + 1)$ are factors of $x^3 + 3x^2 + ax + b$ find a and b.
11. If $x^2 - 5x + 6$ is a factor of $3x^3 + ax^2 + bx + 24$, find a and b.
12. Given that $x^2 - 9$ is a factor of $x^3 + px^2 + qx - 45$ find p and q.
13. If $x^3 + ax^2 + bx + 6$ has $x - 2$ as a factor and leaves a remainder 3 when divided by $x - 3$ find a and b.
14. Show by factor theorem $x - y$, $y - z$ and $z - x$ are the factors of $x^2(y - z) + y^2(z - x) + z^2(x - y)$.
15. Using factor theorem show that $a - b$, $b - c$ and $c - a$ are the factors of $a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$.
16. If $(x - 2)$ is a factor of $x^2 + ax + b$ and $a + b = 1$, find the values of a and b.

5.2.3 Synthetic Division

Let us consider an ordinary long division

$(3x^3 - 5x^2 + 5) \div (x - 2)$

$$
\begin{array}{r|l}
 & 3x^2 + x \; + 2 \\ \hline
x - 2 & 3x^3 - 5x^2 + 0x + 5 \\
 & 3x^3 - 6x^2 \\ \hline
 & \qquad x^2 + 0x \\
 & \qquad x^2 - 2x \\ \hline
 & \qquad\qquad 2x + 5 \\
 & \qquad\qquad 2x - 4 \\ \hline
 & \qquad\qquad\quad\;\; 9 \\ \hline
\end{array}
$$

If we omit all the x's, retaining only the coefficients, this calculation can be abbreviated as follows :

$$
\begin{array}{r|cccc}
 & 3 & 1 & 2 & \\ \hline
-2 & 3 & -5 & 0 & 5 \\
 & \textcircled{3} & -6 & & \\ \hline
 & & 1 & \boxed{0} & \\
 & & \textcircled{1} & -2 & \\ \hline
 & & & 2 & \boxed{5} \\
 & & & \textcircled{2} & -4 \\ \hline
 & & & & 9 \\ \hline
\end{array}
$$

In this, the encircled numbers 3, 1 and 2 are simply repetitions of the numbers directly above them. And the ensquared numbers 0 and 5 are repetitions of numbers in the original. If we omit all these repeated numbers, we get the following array :

$$
\begin{array}{r|cccc}
 & 3 & 1 & 2 & \\ \hline
-2 & 3 & -5 & 0 & 5 \\
 & & -6 & & \\ \hline
 & & 1 & & \\
 & & & -2 & \\ \hline
 & & & 2 & \\
 & & & & -4 \\ \hline
 & & & & 9 \\ \hline
\end{array}
$$

This array can be compressed vertically and we get

	3	1	2		→ top row
–2)	3	–5	0	5	
		–6	–2	–4	
	(3)	1	2	9	

The extra encircled 3 has been copied from the top row. With this extra number, the bottom row actually duplicates the top row, plus one additional number, the remainder. Therefore, we can omit the top row entirely, and we get

–2)	3	–5	0	5	→ top row
		–6	–2	–4	
	3	1	2	9	

We observe that in this array, each element in the middle row is formed by multiplying the preceeding element in the bottom row by –2. We subtract the middle row from the top row.

Instead, if we drop the negative sign in the divisior, we have exactly the method of synthetic division. Here, we are adding the middle row to the top row. This helps us to cut down arithmetic errors. Thus by synthetic division we have

2	3	–5	0	5
		6	2	4
	3	1	2	9

Thus the quotient is $3x^2 + x + 2$ and the remainder is 9.

Note : The remainder must have degree less than the divisor. In the present case, the divisor is of the form $(x - c)$ has degree 1. Therefore the remainder must be a polynomial of degree zero - that is, *the remainder must always be just a constant.*

Example 5.22 : Find the quotient and remainder when $x^3 + x^2 - 7x - 3$ is divided by $x - 3$.

Solution :

3	1	1	–7	–3	
		3	12	15	
	1	4	5	12	⇒ Remainder

The remainder is 12 and the quotient is $x^2 + 4x + 5$.

Example 5.23 : Find the quotient and the remainder when $8x^4 - 2x^2 + 6x - 5$ is divided by $2x + 1$.

The dividend is $8x^4 + 0x^3 - 2x^2 + 6x - 5$ and the divisor is $2x + 1 = 2\,(x + ½)$

So we divide $8x^4 + 0x^3 - 2x^2 + 6x - 5$ by $x + ½$ first and then we divide the resulting quotient by 2.

–½	8	0	–2	6	–5	
		–4	2	0	–3	
	8	–4	0	6	–8	⇒ Remainder

The quotient is $½\,(8x^3 - 4x^2 + 6) = 4x^3 - 2x^2 + 3$ and the remainder is –8.

Example 5.24. If the quotient on dividing $2x^4 - 7x^3 - 13x^2 + 63x - 48$ by $x - 1$ is $2x^3 + ax^2 + bx + 45$, find a and b.

Solution :

1	2	–7	–13	63	–48	
		2	–5	–18	45	
	2	–5	–18	45	–3	⇒ Remainder

The quotient is $2x^3 - 5x^2 - 18x + 45$

But the given quotient is $2x^3 + ax^2 + bx + 45$

Hence $a = -5$ and $b = -18$.

Example 5.25 : Show that $(y + 2)$ is a factor of $y^3 - 2y^2 - 29y - 42$

What are the other factors ?

Solution :

–2	1	–2	–29	–42
		–2	8	42
	1	–4	–21	0

Quotient is $y^2 - 4y - 21$

Since the remainder is 0, $(y + 2)$ is a factor of $y^3 - 2y^2 - 29y - 42$.

To get the other factors we have to factorise $y^2 - 4y - 21$

(i.e.,) $y^2 - 4y - 21 = (y - 7)\,(y + 3)$.

$\therefore$ the other factors are $(y - 7)$ and $(y + 3)$.

EXERCISE 5.5

1. Find the quotient and the remainder when $x^3 + 2x^2 - x - 4$ is divided by $x + 2$.
2. Find the quotient and the remainder when $4x^4 - 3x^3 - x + 7$ is divided by $2x - 1$.
3. If the quotient on dividing $3x^3 - 2x^2 + 7x - 5$ by $x + 3$ is $3x^2 - 11x + a$, find a.
4. Show that $3z + 10$ is a factor of $9z^3 - 27z^2 - 100z + 300$. Find also the other factors.
5. If the quotient on dividing $x^4 + 10x^3 + 35x^2 + 50x + 29$ by $x + 4$ is $x^3 + ax^2 + bx + 6$, find a and b.
6. If the quotient on dividing $2y^4 + y^3 - 14y^2 - 19y - 6$ by $2x + 1$ is $y^3 + py^2 + qy - 6$, find p and q.
7. Show that $(y + 2)$ is a factor of $2y^3 - 7y^2 - 10y + 24$ and also find the other factors.

5.2.4 Factorisation

Let us consider a polynomial $p(x) = ax^4 + bx^3 + cx^2 + dx + e$.

(i) If we divide this by $x - 1$, the remainder is $p(1)$

$\therefore$ The remainder is $a + b + c + d + e$.

If $x - 1$ is a factor of $p(x)$, then $p(1) = 0$

That is $a + b + c + d + e = 0$

In a polynomial containing different powers of x, if the algebraic sum of the coefficients of all powers of x is equal to zero, then $(x - 1)$ is a factor.

(ii) If we divide $p(x)$ by $(x + 1)$, the remainder is $p(-1)$.

$\therefore$ The remainder $p(-1) = a - b + c - d + e$

If $x + 1$ is a factor of $p(x)$ then $a - b + c - d + e = 0$

That is, $a + c + e = b + d$.

In a polynomial, $p(x)$ containing different powers of x, if the algebraic sum of the coefficients of the even powers of x is equal to the sum of the coefficients of its odd powers, then $x + 1$ is a factor of $p(x)$.

Note : If $a + c + e = b + d = 0$, then $x^2 - 1$ is a factor.

(a) Consider $x^3 - 6x^2 + 11x - 6$

$1 - 6 + 11 - 6 = 0$. Hence, $x - 1$ is a factor.

Also $1 + 11 \neq -6 - 6$, $\therefore$ $x + 1$ is not a factor.

(b) Consider $x^3 + 6x^2 + 11x + 6$

$1 + 6 + 11 + 6 \neq 0$ $\therefore$ $x - 1$ is not a factor.

$1 + 11 = 6 + 6 = 12$ $\therefore$ $x + 1$ is a factor.

(c) Consider $x^4 + 2x^3 + 2x^2 - 2x - 3$

(i) $1 + 2 + 2 - 2 - 3 = 0$ $\therefore$ $x - 1$ is a factor.

(ii) $1 + 2 - 3 = 2 - 2 = 0$ $\therefore$ $x + 1$ is a factor.

$\therefore$ $x^2 - 1$ is a factor .

Example 5.26 : Resolve into factors $x^3 - 2x^2 - 5x + 6$

Sum of the coefficients = $1 - 2 - 5 + 6 = 0$ $\therefore$ $x - 1$ is a factor.

1	1	–2	–5	6
		1	–1	–6
	1	–1	–6	0

The others are given by factorizing $x^2 - x - 6$

$$\text{Now } x^2 - x - 6 = x^2 - 3x + 2x - 6$$

$$= (x - 3)(x + 2)$$

$$\therefore x^3 - 2x^2 - 5x + 6 = (x - 1)(x + 2)(x - 3)$$

Example 5.27 : Factorize : $x^3 + 6x^2 + 11x + 6$

$1 + 6 + 11 + 6 \neq 0$ $\therefore$ $x - 1$ is not a factor.

$1 + 11 = 12$; $6 + 6 = 12$ $\therefore$ $x + 1$ is a factor.

–1	1	6	11	6	
		–1	–5	–6	
–2	1	5	6	0	(by trial and error method $x + 2$ is a factor)
		–2	–6		
	1	3	0		

Now $x + 3$ is also a factor.

$\therefore$ $x^3 + 6x^2 + 11x + 6 = (x + 1)(x + 2)(x + 3)$

Example 5.28 : Factorize $2x^3 - 3x^2 - 3x + 2$

$2 - 3 - 3 + 2 \neq 0$ $\therefore$ $x - 1$ is not a factor.

$2 - 3 = -3 + 2 = -1$ $\therefore$ $x + 1$ is a factor.

–1	2	–3	–3	2	
		–2	5	–2	
2	2	–5	2	0	(by using $x - 2$ as a factor)
		4	–2		
	2	–1	0		

$\therefore 2x - 1$ is also a factor.

Thus $2x^3 - 3x^2 - 3x + 2 = (x + 1)(x - 2)(2x - 1)$

EXERCISE 5.6

Factorize each of the following fully.

(1) $x^3 - 4x^2 + 5x - 2$ (2) $x^3 + 9x^2 + 23x + 15$ (3) $x^3 - 2x^2 - 5x + 6$

(4) $2x^4 + 7x^3 + x^2 - 7x - 3$ (5) $x^3 + x^2 + x - 14$ (6) $m^3 + 3m^2 - 4m - 12$

(7) $m^3 - 2m^2 - 4m + 8$ (8) $a^3 - 5a^2 - 2a + 24$ (9) $x^4 - 5x^2 + 4$

5.3 G.C.D and L.C.M of Polynomials

5.3.1 Greatest Common Divisor (GCD) (or) Highest Common Factor (HCF)

A factor which cannot be resolved into any other factor is called an **Elementary Factor.** The elementary factors of a^2b^2 are a and b, because a and b cannot be resolved into any other factor.

A factor that divides two or more expressions is said to be a **Common Factor** of those expressions. Thus a, b and ab are common factors of a^2b and ab^2c.

THE HIGHEST COMMON FACTOR (H.C.F) :

There may be more than one common factor of two or more expressions. Of these common factors, the factor which is of the highest power is the **Highest Common Factor** of these expressions.

For example, the common factors of $2a^2b^3c^2$, $3a^4b^2c^3$ and $4a^5b^3c^2$ are $a, b, c, a^2, b^2, c^2, ab, ac, bc, a^2b, a^2c, b^2c, bc^2, abc, a^2bc, ab^2c, abc^2$ and $a^2b^2c^2$. Of these common factors, $a^2b^2c^2$ is of the highest power. So, $a^2b^2c^2$ is the **H.C.F (or G.C.D)**.

To find the H.C.F (G.C.D) by Factorization :

(i) The given expressions should first be resolved into elementary factors.

(ii) The H.C.F will be the product of the common elementary factors of the highest powers that divide each of these expressions.

(iii) If the expressions have numerical coefficients, their arithmetical G.C.D will be the coefficient of the H.C.F.

Example 5.29 : Find the HCF of $21a^4x^3y$, $35a^2x^4y$, $28a^3xy^4$.

Solution :

$$21a^4x^3y = 7 \times 3 \times a \times a \times a \times a \times x \times x \times x \times y$$

$$35a^2x^4y = 7 \times 5 \times a \times a \times x \times x \times x \times x \times y$$

$$28a^3xy^4 = 7 \times 4 \times a \times a \times a \times x \times y \times y \times y \times y$$

GCD $= 7 \times a \times a \times x \times y$

GCD $= 7a^2xy$

Example 5.30 : Find the GCD of $15x^4y^3z^2$, $12x^2yz^2$

Solution : $15x^4y^3z^2 = 5 \times 3 \times x^4y^3\, z^2$

$12x^2yx^2 = 4 \times 3 \times x^2yz^2$

GCD $= 3x^2yz^2$

Example 5.31 : Find the GCD if $f(x) = (a-1)^4\,(a+3)^3$ and $g(x) = (a-2)\,(a-1)^3\,(a+3)^4$

$f(x) = (a-1)^4\,(a+3)^3$
$g(x) = (a-2)\,(a-1)^3\,(a+3)^4$

GCD $= (a-1)^3\,(a+3)^3$

Example 5.32 : Find the GCD of (y^3+1) and (y^2-1)

Solution : $y^3 + 1 = (y+1)\,(y^2 - y + 1)$

$y^2 - 1 = (y+1)\,(y-1)$

GCD $= y + 1$

Example 5.33 : Find the GCD of the polynomials $2x^2 - 18$ and $x^2 - 2x - 3$.

Solution : $2x^2 - 18 = 2(x^2 - 9) = 2(x-3)\,(x+3)$

$x^2 - 2x - 3 = (x-3)\,(x+1)$

$\therefore$ G.C.D $= x - 3$

Example 5.34 : Find the GCD of $2x^2 - x - 1$ and $4x^2 + 8x + 3$.

Solution : $2x^2 - x - 1 = 2x^2 - 2x + x - 1$

$= 2x\,(x-1) + 1\,(x-1)$

$$= (2x + 1)(x - 1) \quad \text{... (1)}$$

$$4x^2 + 8x + 3 = 4x^2 + 6x + 2x + 3$$

$$= 2x(2x + 3) + 1(2x + 3)$$

$$= (2x + 1)(2x + 3) \quad \text{... (2)}$$

$\therefore$ from (1) and (2) GCD $= 2x + 1$

Example 5.35 : Find the HCF of $ax^2 + 2a^2x + a^3$, $2ax^2 - 4a^2x - ba^3$, $3(ax + a^2)^2$

Solution : $ax^2 + 2a^2x + a^3 = a(x^2 + 2ax + a^2)$

$$= a(x + a)^2 \quad \text{... (1)}$$

$$2ax^2 - 4a^2x - 6a^3 = 2a(x^2 - 2ax - 3a^2)$$

$$= 2a(x^2 - 3ax + ax - 3a^2)$$

$$= 2a[x(x - 3a) + a(x - 3a)]$$

$$= 2a[(x + a)(x - 3a)] \quad \text{... (2)}$$

$$3(ax + a^2)^2 = 3[a(x + a)]^2$$

$$= 3a^2(x + a)^2 \quad \text{... (3)}$$

From (1), (2) and (3) HCF $= a(x + a)$

Example 5.36 : Find the GCD of $24(6x^4 - x^3 - 2x^2)$ and $20(2x^6 + 3x^5 + x^4)$

Solution : $24(6x^4 - x^3 - 2x^2) = 2^3 \times 3 \times x^2(6x^2 - x - 2)$

$$= 2^3 \times 3 \times x^2(6x^2 - 4x + 3x - 2)$$

$$= 2^3 \times 3 \times x^2[2x(3x - 2) + 1(3x - 2)]$$

$$= 2^3 \times 3 \times x^2(3x - 2)(2x + 1) \quad \text{... (1)}$$

$$20(2x^6 + 3x^5 + x^4) = 2^2 \times 5 \times x^4(2x^2 + 3x + 1)$$

$$= 2^2 \times 5 \times x^4(2x^2 + 2x + x + 1)$$

$$= 2^2 \times 5 \times x^4[2x(x + 1) + 1(x + 1)]$$

$$= 2^2 \times 5 \times x^4(2x + 1)(x + 1) \quad \text{... (2)}$$

$\therefore$ from (1) and (2) GCD $= 2^2 \times x^2(2x + 1) = 4x^2(2x + 1)$

Example 5.37 : Find the HCF of $x^3 + x^2 + x + 1$ and $x^4 - 1$

Solution : $x^3 + x^2 + x + 1 = x^2(x + 1) + 1(x + 1)$

$$= (x^2 + 1)(x + 1) \quad \text{... (1)}$$

$$x^4 - 1 = (x^2)^2 - 1^2$$
$$= (x^2 - 1)(x^2 + 1)$$
$$= (x + 1)(x - 1)(x^2 + 1) \quad \text{... (2)}$$

$\therefore$ from (1) and (2) HCF $= (x + 1)(x^2 + 1)$

EXERCISE 5.7

I. Find the GCD of the following :

1. $4ab^2, 2a^2b$
2. $6xy^2z, 8x^2y^3z^2$
3. $5a^3b^3, 15abc^2$
4. $49ax^2, 63ay^2, 56az^2$
5. $8a^2x, 6abxy, 10abx^3y^2$
6. $25xy^2z, 100x^2yz, 125xy$

II. Find the HCF of the following:

1. $(x - 2)^2(x + 3)(x - 4), (x - 2)(x + 2)(x - 3)$
2. $(2x - 7)(3x + 4), (2x - 7)^2(x + 3)$
3. $(x - 1)(x + 1)^3, (x - 1)^3(x + 1)$
4. $(x + 4)^2(x - 3)^3, (x - 1)(x + 4)(x - 3)^2$
5. $24(x - 3)(x - 2)^2, 15(x - 2)(x - 3)^3$
6. $a^2 + ab, a^2 - b^2$
7. $(x + y)^2, x^2 - y^2$
8. $x^2 + 3x + 2, x^2 - 4$
9. $xy - y, x^4y - xy$
10. $x^2 + 6x + 5, x^2 + 8x + 15$
11. $x^2 - 81, x^2 + 6x - 27$
12. $x^2 - 17x + 66, x^2 + 5x - 66$
13. $2x^2 - 7x + 3, 3x^2 - 7x - 6$
14. $12x^2 + x - 1, 15x^2 + 8x + 1$
15. $x^3 + 3x^2 - 8x - 24, x^3 + 3x^2 - 3x - 9$
16. $(x - 4)(x^2 - 11x + 30), (x^6 - 5x^5 + x - 5)$
17. $(x - 1)^3, (x^4 - x^3 + 2x - 2)$
18. $x^3 + 8x^2 - x - 8, x^3 + x^2 - x - 1$
19. $(x - 3)^2, x^2 - 9, x^2 - x - 6$
20. $2(x^2 - 1), 3(x^3 - 1), 4(x^2 - 5x + 4)$

5.3.2 Least Common Multiple (L.C.M.)

If a quantity is exactly divisible by another, the former is called a **multiple** of the latter. For example, a^3 is exactly divisible by a^2 and a. So, a^3 is a multiple of both a^2 and a.

Those expressions, which are exactly divisible by two or more other expressions are the **common multiples** of the latter. For example, ab, a^2b, a^2b^2 are exactly divisible by a and b. So, they are the common multiples of a and b.

The Least Common Multiple of two or more expressions is the expression of lowest dimensions (or powers) which is exactly divisible by each of them.

For example, a^2b^2, a^3b^3 and a^3b^2 are common multiples of a^2 and b^2 and here a^2b^2 is of the lowest power. So, a^2b^2 is the lowest common multiple here. The lowest common multiple is briefly written as **L.C.M.**

To find the L.C.M by Factorization :

(i) Each expression is first to be resolved into factors.

(ii) The product of the factors having the highest powers in those factors will be the L.C.M.

(iii) The numerical coefficient of the L.C.M will be the arithmetical L.C.M of the numerical coefficients of the given expressions.

Example 5.38 : Find the LCM of (*i*) x^3y^2 and xyz (*ii*) 5 a^2bc^3, $4ab^2c$ *(iii)* $3x^2 - 27$ and $x^2 + x - 12$ (*iv*) $x^3 + y^3$, $x^3 - y^3$, $x^4 + x^2y^2 + y^4$.

Solution : (*i*) By inspection,

$$\text{LCM} = x^3 y^2 z.$$

(*ii*) By inspection, $\text{LCM} = 5 \times 4 \times a^2 \times b^2 \times c^3$

$$= 20a^2 b^2 c^3$$

(*iii*) $3x^2 - 27 = 3(x^2 - 9) = 3\,(x + 3)\,(x - 3)$

$$x^2 + x - 12 = (x + 4)\,(x - 3)$$

Product of the numerical factors = 3 × 1 = 3

Common factors to both polynomials = $x - 3$

Remainingfactors are $(x + 3)$ and $(x + 4)$

$$\therefore \text{ LCM} = 3(x - 3)\,(x + 3)\,(x + 4)$$

(*iv*) $x^3 + y^3 = (x + y)\,(x^2 - xy + y^2)$

$$x^3 - y^3 = (x - y)\,(x^2 + xy + y^2)$$

$$x^4 + x^2y^2 + y^4 = (x^2 + xy + y^2)\,(x^2 - xy + y^2)$$

$$\therefore \text{ LCM} = (x + y)\,(x - y)\,(x^2 + xy + y^2)\,(x^2 - xy + y^2)$$

$$= [(x + y)\,(x^2 - xy + y^2)]\,[(x - y)\,(x^2 + xy + y^2)]$$

$$= (x^3 + y^3)\,(x^3 - y^3)$$

$$\therefore \text{ LCM} = x^6 - y^6$$

Example 5.39 : Find the LCM of $2(x^3 + x^2 - x - 1)$ and $3(x^3 + 3x^2 - x - 3)$

Solution : $2(x^3 + x^2 - x - 1) = 2[x^2\,(x + 1) - (x + 1\,)]$

$$= 2\,[(x^2 - 1)\,(x + 1)]$$

$$3(x^3 + 3x^2 - x - 3) = 3[x^2\,(x + 3) - 1\,(\,x + 3)]$$

$$= 3\,(x^2 - 1)\,(x + 3)$$

Product of numerical factors = 2 × 3 = 6

Product of common factors = $(x^2 - 1)$

Product of remaining factors = $(x + 1)(x + 3)$

$$\therefore \text{ LCM } = 6(x^2 - 1)(x + 1)(x + 3)$$

EXERCISE 5.8

1. Find the LCM of the following :
   (i) $3a^4 b^2 c^3, 5a^2 b^3 c^5$
   (ii) $15x^3y^3z, 25xy^3 z^2$
   (iii) $15p^3q^4, 20m^2p^2q^3, 30mp^3$
2. Find the LCM of the following :
   (i) $x^2 - 1, (x - 1)^2$
   (ii) $x^3 - y^3, x^2 - y^2$
   (iii) $x^2 - 10x + 24, x^2 - 11x + 30$
   (iv) $13(x - 1)(x - 2)^2, 7(x - 2)^2 (x + 3)^2, (x - 1)^2 (x + 3)$
   (v) $x^2 - 12x + 35, x^2 - 8x + 7, x^3 - 5x^2 - x + 5$
   (vi) $6x^2 - x - 1, 3x^2 + 7x + 2, 2x^2 + 3x - 2$
   (vii) $x^3 + x^2 + x + 1, x^3 + 2x^2 + x + 2$

5.3.3 Relation between H.C.F and L.C.M. of two polynomials

We know that the product of two natural numbers is the product of their G.C.D. and L.C.M.

Consider 15 and 20

Their G.C.D is 5 and L.C.M is 60

Now, 15 × 20 = 5 × 60.

This concept holds good in the case of two polynomials also.

Consider $P(x) = x^2 + 3x + 2$ and $Q(x) = x^2 - x - 6$

Their G.C.D. is $(x + 2)$ and L.C.M. is $(x + 2)(x + 1)(x - 3)$

Now, $P(x) \times Q(x) = (x^2 + 3x + 2)(x^2 - x - 6)$

$= (x + 1)(x + 2)(x + 2)(x - 3)$

$= (x + 1)(x + 2)^2 (x - 3)$

$$\text{G.C.D} \times \text{L.C.M} = (x + 2)(x + 2)(x + 1)(x - 3)$$

$$= (x + 1)(x + 2)^2 (x - 3)$$

$$\text{Hence, } P(x) \times Q(x) = \text{G.C.D.} \times \text{L.C.M}$$

Example 5.40 : The g.c.d. of $x^4 + 3x^3 + 5x^2 + 26x + 56$ and $x^4 + 2x^3 - 4x^2 - x + 28$ is $x^2 + 5x + 7$. Find their l.c.m.

Solution : We know, $P(x) \times Q(x) = \text{G.C.D} \times \text{L.C.M}$.

$$\text{L.C.M} = \frac{P(x) \times Q(x)}{\text{G.C.D}}$$

$$= \frac{(x^4+3x^3+5x^2+26x+56)\,(x^4+2x^3-4x^2-x+28)}{(x^2+5x+7)}$$

Both P(x) and Q(x) is divisible by $x^2 + 5x + 7$ (as this is their G.C.D). Let us divide P(x) and get the quotient.

			1	–2	8		
1	5	7	1	3	5	26	56
			1	5	7		
				–2	–2	26	
				–2	–10	–14	
					8	40	56
					8	40	56
						0	

$\therefore$ Their L.C.M is $(x^2 - 2x + 8)(x^4 + 2x^3 - 4x^2 - x + 28)$

EXERCISE 5.9

1. If $(x + 3)(x - 2)$ is the g.c.d. of $f(x) = (x + 3)(2x^2 - 3x + a)$ and $g(x) = (x - 2)(3x^2 + 7x - b)$ find the values of a and b.
2. For what value of k, the G.C.D. of $[x^2 + x - (2k + 2)]$ and $(2x^2 + kx - 12)$ is $(x + 4)$?
3. Find the values of a and b in each of the following such that the polynomials P(x) and Q (x) have the given G.C.D (H.C.F)

	P(*x*)	**Q(*x*)**	**G.C.D**
(i)	$(x^2 + 3x + 2)(x^2 + x + a)$	$(x^2 - 3x + 2)(x^2 - 3x + b)$	$(x + 1)(x - 2)$
(ii)	$(x^2 + 3x + 2)(x^2 - 4x + a)$	$(x^2 - 6x + 9)(x^2 + 4x + b)$	$(x + 2)(x - 3)$

4. Find the other polynomial Q(x), given that L.C.M and G.C.D and one polynomial P(x) respectively.

   (i) $(x + 1)^2 (x + 2)^2$; $(x + 1) (x + 2)$; $(x + 1)^2 (x + 2)$

   (ii) $(x^4 - y^4) (x^4 + x^2y^2 + y^4)$; $x^2 - y^2$; $x^4 - y^4$

5. Find the L.C.M of each of the following polynomials given their G.C.D.

   (i) $x^4 + 3x^3 + 6x^2 + 5x + 3$; $x^4 + 2x^2 + x + 2$; G.C.D in $x^2 + x + 1$

   (ii) $2x^3 + 15x^2 + 2x - 35$; $x^3 + 8x^2 + 4x - 21$; G.C.D in $(x + 7)$

5.4 SIMPLIFICATION OF RATIONAL EXPRESSIONS

Rational Expression : An expression of the form $\frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are two polynomials over the set of real numbers and $q(x) \neq 0$ is called a rational expression.

5.4.1 Simplification of rational expressions :

A rational expression $\frac{p(x)}{q(x)}$ can be reduced to its simplest form by dividing both numerator $p(x)$ and denominator $q(x)$ by the G.C.D. of $p(x)$ and $q(x)$.

In other words, the rational expression $\frac{p(x)}{q(x)}$ is said to be in its simplest form if the G.C.D. of $p(x)$ and $q(x)$ is one.

Example 5.41 : Simplify : $\frac{12a^3b^4}{20a^4b}$

Solution :

$$\frac{12a^3b^4}{20a^4b} = \frac{2^2 \times 3 \times a^3 \times b^4}{2^2 \times 5 \times a^4 \times b} = \frac{3\,b^3}{5\,a}\,\frac{2^2}{2^2}\,\frac{a^3}{a^3}\,\frac{b}{b} = \frac{3b^3}{5a}$$

Example 5.42 : Simplify : $\frac{x^3 - 5x^2}{3x^3 + 2x^4}$

Solution :

$$\frac{x^3 - 5x^2}{3x^3 + 2x^4} = \frac{x^2\ (x - 5)}{x^2\ (3x + 2x^2)} = \frac{x - 5}{3x + 2x^2}$$

Example 5.43 : Simplify : $\dfrac{x^2 + x - 6}{x^2 + 4x + 3}$

Solution : $\dfrac{x^2 + x - 6}{x^2 + 4x + 3} = \dfrac{(x-2)\,(x+3)}{(x+1)\,(x+3)} = \dfrac{x-2}{x+1}$

Example 5.44 : Simplify : $\dfrac{a+b}{a^3 + b^3}$

Solution : $\dfrac{a+b}{a^3 + b^3} = \dfrac{a+b}{(a+b)\,(a^2 - ab + b^2)} = \dfrac{1}{a^2 - ab + b^2}$

Example 5.45 : Simplify : $\dfrac{6x^2 - 5x + 1}{9x^2 + 12x - 5}$

Solution : $\dfrac{6x^2 - 5x + 1}{9x^2 + 12x - 5} = \dfrac{6x^2 - 3x - 2x + 1}{9x^2 + 15x - 3x - 5}$

$$= \dfrac{3x(2x-1) - 1(2x-1)}{3x(3x+5) - 1(3x+5)}$$

$$= \dfrac{(3x-1)\,(2x-1)}{(3x-1)\,(3x+5)} = \dfrac{2x-1}{3x+5}$$

Example 5.46 : Simplify : $\dfrac{(x-8)\,(x^2+5x-50)}{(x^2-13x+40)(x+10)}$

Solution : $\dfrac{(x-8)\,(x^2+5x-50)}{(x^2-13x+40)(x+10)} = \dfrac{(x-8)\,(x+10)\,(x-5)}{(x-8)\,(x-5)\,(x+10)} = 1$

EXERCISE 5.10

Simplify :

1. $\dfrac{75x^5}{25x^3}$ 2. $\dfrac{24x^3y^2}{-6x^2y^4z}$ 3. $\dfrac{8y^2-12y}{4y}$

4. $\dfrac{x^2 - 1}{xy + y}$ 5. $\dfrac{6x^2 + 9x}{3x^2 - 12x}$ 6. $\dfrac{x^2 + 7x + 12}{x^2 + 4x + 3}$

7. $\dfrac{x^2 - 6x + 8}{x^2 - 3x + 2}$ 8. $\dfrac{4x^2 - 13x + 3}{4x - 1}$ 9. $\dfrac{(x-1)(x-2)(x^2 - x - 72)}{(x-9)(x^2 + x - 2)}$

10. $\dfrac{2x^4 - 162}{(x^2 + 9)(2x - 6)}$ 11. $\sqrt{\dfrac{(x^2 + 3x + 2)(x^2 + 5x + 6)}{x^2 (x^2 + 4x + 3)}}$

12. $\dfrac{(x-1)(x-2)(x^2 - 9x + 14)}{(x-7)(x^2 - 3x + 2)}$

5.4.2 Multiplication and Division of Rational Expressions

Multiplication : If $\dfrac{p(x)}{q(x)}$ and $\dfrac{g(x)}{h(x)}$ are two rational expressions then their product is

$$\frac{p(x)}{q(x)} \times \frac{g(x)}{h(x)} = \frac{p(x).g(x)}{q(x).h(x)}$$

The resulting expression is then reduced to its simplest form.

In other words, the product of two rational expressions is the product of their numerators divided by the product of their denominators and the resulting expression is reduced to the simplest form.

Division : If $\dfrac{p(x)}{q(x)}$ and $\dfrac{g(x)}{h(x)}$ are two rational expressions, then their quotient is

$$\frac{p(x)}{q(x)} \div \frac{g(x)}{h(x)} = \frac{p(x)}{q(x)} \times \frac{h(x)}{g(x)} = \frac{p(x).h(x)}{q(x).g(x)}$$

The resulting expression is then reduced to its simplest form.

Example 5.47 : Multiply : $\dfrac{x^3}{3y^2}$ by $\dfrac{18y}{x^4}$

Solution : $\dfrac{x^3}{3y^2} \times \dfrac{18y}{x^4} = \dfrac{x^3 \times 18y}{3y^2 \times x^4} = \dfrac{6}{xy}$

Example 5.48 : Multiply : $\frac{a^3 b^2}{a-1}$ by $\frac{a^2-1}{a^2 b^3}$

Solution :

$$\frac{a^3 b^2}{a-1} \times \frac{a^2-1}{a^2 b^3} = \frac{a^3 b^2 \; (a+1)\,(a-1)}{(a-1)\;a^2\;b^3}$$

$$= \frac{a\,(a+1)}{b}$$

Example 5.49 : If P = $\frac{4x}{x^2-1}$ and Q = $\frac{x+1}{x-1}$, find PQ.

Solution :

$$PQ = \frac{4x}{x^2-1} \times \frac{x+1}{x-1}$$

$$= \frac{4\,x\;(x+1)}{(x+1)\,(x-1)\;(x-1)}$$

$$= \frac{4\,x}{(x-1)^2}$$

Example 5.50 : Simplify : $\frac{2x^2+3x+1}{3x^2+4x+1} \times \frac{4x^2+5x+1}{5x^2+6x+1} \times \frac{15x^2+8x+1}{8x^2+6x+1}$

Solution : $\frac{2x^2+3x+1}{3x^2+4x+1} \times \frac{4x^2+5x+1}{5x^2+6x+1} \times \frac{15x^2+8x+1}{8x^2+6x+1}$

$$= \frac{(2x^2+2x+x+1)(4x^2+4x+x+1)(15x^2+5x+3x+1)}{(3x^2+3x+x+1)(5x^2+5x+x+1)(8x^2+4x+2x+1)}$$

$$= \frac{[2x(x+1)+1(x+1)]\,[4x(x+1)+1(x+1)]\;[5x(3x+1)+1(3x+1)]}{[3x(x+1)+1(x+1)]\,[5x(x+1)+1(x+1)]\;[4x(2x+1)+1(2x+1)]}$$

$$= \frac{(2x+1)\ (x+1)\ (4x+1)\ (x+1)\ (5x+1)\ (3x+1)}{(3x+1)\ (x+1)\ (5x+1)\ (x+1)\ (4x+1)\ (2x+1)} = 1$$

Example 5.51 : Divide : $\frac{x^2+x-6}{x^2-x-6}$ by $\frac{x^2-9}{x^2-4}$

Solution : $$\frac{x^2+x-6}{x^2-x-6} \div \frac{x^2-9}{x^2-4} = \frac{x^2+x-6}{x^2-x-6} \times \frac{x^2-4}{x^2-9}$$

$$= \frac{(x+3)\ (x-2)\ \ (x+2)\ (x-2)}{(x-3)\ (x+2)\ \ (x+3)\ (x-3)}$$

$$= \frac{(x-2)^2}{(x-3)^2}$$

Example 5.52 : Divide : $\frac{3x^2-x-4}{9x^2-16}$ by $\frac{4x^2-4}{3x^2-2x-1}$

Solution : $$\frac{3x^2-x-4}{9x^2-16} \div \frac{4x^2-4}{3x^2-2x-1}$$

$$= \frac{3x^2-x-4}{9x^2-16} \times \frac{3x^2-2x-1}{4x^2-4}$$

$$= \frac{(3x^2-4x+3x-4)\ (3x^2-3x+x-1)}{(3x+4)(3x-4)\ 4(x^2-1)}$$

$$= \frac{[x(3x-4)+1(3x-4)]\ [3x\ (x-1)+1(x-1)]}{(3x+4)(3x-4)\ 4(x+1)(x-1)}$$

$$= \frac{(3x+1)\ \ \ (3x-4)\ (x+1)\ (x-1)}{4(3x+4)\ \ (3x-4)\ (x+1)\ (x-1)}$$

$$= \frac{3x+1}{4(3x+4)}$$

EXERCISE 5.11

I. Simplify :

1. $\frac{11}{5mn^2} \times \frac{15mn}{44}$

2. $\frac{32a^3b}{15c} \times \frac{45c^2}{8a^2b}$

3. $\frac{2x+2y}{5} \times \frac{10}{x+y}$

4. $\frac{y^2-2y}{y+2} \times \frac{3y+6}{y-2}$

5. If P = $\frac{x^2+3}{x^2-1}$ and Q = $\frac{x-1}{2x}$, find PQ.

6. $\frac{x^3-8}{x^2-4} \times \frac{x^2+6x+8}{x^2-2x+1}$

7. $\frac{x^2+3x+2}{x^2-4x-12} \times \frac{x^2-7x+6}{x^2-4}$

8. $\frac{x^2-y^2}{x^2+2xy+y^2} \times \frac{xy+y^2}{x^2-xy}$

9. $\frac{x^4-8x}{2x^2+5x-3} \times \frac{2x-1}{x^2+2x+4} \times \frac{x+3}{x^2-2x}$

10. $\frac{6x^2-x-2}{8x^2+6x+1} \times \frac{12x^2+7x-12}{9x^2+6x-8} \times \frac{12x^2-13x-4}{12x^2-25x+12}$

II. Divide :

1. $\frac{14x^2}{9y^2} \div \frac{25x^2}{36y^2}$

2. $\frac{75m^2}{14n^2} \div \frac{25mn}{49}$

3. $\frac{t-2}{4s} \div \frac{t^2-t-6}{12s^2}$

4. $\frac{x+y}{x^2-y^2} \div \frac{x^2-xy}{x^2-2xy+y^2}$

5. $\frac{x^2-6x+9}{x^2-4x+4} \div \frac{x^2-5x+6}{x^2-x-2}$

6. $\frac{x^2-7x+12}{x^2-16} \div \frac{x^2-2x-3}{x^2-2x-24}$

7. If P = $\frac{x^2-36}{x^2-49}$ and Q = $\frac{x+6}{x+7}$ find the value of $\frac{P}{Q}$.

8. If A = $\frac{x^2-5x+6}{x^2-9x+20}$ and B = $\frac{x^2-3x+2}{x^2-5x+4}$ find the value of A ÷ B.

9. $\dfrac{2x^2+5x-3}{2x^2+7x+6} \div \dfrac{2x^2+5x-3}{2x^2-x-6}$

10. $\dfrac{2x^2+7x-4}{3x^2-13x+4} \div \dfrac{4x^2-1}{6x^2+x-1}$

5.4.3 Addition and Subtraction of Rational Expressions.

Addition of Rational Expressions :

If $\dfrac{p(x)}{q(x)}$ and $\dfrac{g(x)}{h(x)}$ are two rational expressions then their sum is

$$\frac{p(x)}{q(x)} + \frac{g(x)}{h(x)} = \frac{p(x).h(x)+q(x).g(x)}{q(x).h(x)}$$

If $\dfrac{p(x)}{q(x)}$ and $\dfrac{g(x)}{q(x)}$ are two rational expressions then their sum is

$$\frac{p(x)}{q(x)} + \frac{g(x)}{q(x)} = \frac{p(x)+g(x)}{q(x)}$$

Subtraction of Rational Expressions :

If $\dfrac{p(x)}{q(x)}$ and $\dfrac{g(x)}{h(x)}$ are two rational expressions then their difference is

$$\frac{p(x)}{q(x)} - \frac{g(x)}{h(x)} = \frac{p(x).h(x)-q(x).g(x)}{q(x).h(x)}$$

If $\dfrac{p(x)}{q(x)}$ and $\dfrac{g(x)}{q(x)}$ are two rational expressions then their difference is

$$\frac{p(x)}{q(x)} - \frac{g(x)}{q(x)} = \frac{p(x)-g(x)}{q(x)}$$

Example 5.53 : Simplify : $\dfrac{x+3}{x+2} + \dfrac{x+1}{x-2}$

Solution : $\dfrac{x+3}{x+2} + \dfrac{x+1}{x-2} = \dfrac{(x+3)(x-2)+(x+1)(x+2)}{(x+2)(x-2)}$

$$= \frac{x^2+3x-2x-6+x^2+x+2x+2}{(x+2)(x-2)}$$

$$= \frac{2x^2+4x-4}{x^2-4} = \frac{2(x^2+2x-2)}{x^2-4}$$

Example 5.54 : Simplify : $\frac{x^2-x-6}{x^2-9} + \frac{x^2+2x-24}{x^2-x-12}$

Solution : $\frac{x^2-x-6}{x^2-9} + \frac{x^2+2x-24}{x^2-x-12}$

$$= \frac{(x-3)(x+2)}{(x-3)(x+3)} + \frac{(x+6)(x-4)}{(x+3)(x-4)} = \frac{x+2}{x+3} + \frac{x+6}{x+3}$$

$$= \frac{x+2+x+6}{x+3} = \frac{2x+8}{x+3} = \frac{2(x+4)}{x+3}$$

Example 5.55. : Simplify : (i) $\frac{x}{x+y} - \frac{y}{x-y}$

(ii) $\frac{1}{x^2-5x+6} + \frac{1}{x^2-3x+2} + \frac{2}{x^2-8x+15}$

(iii) $\frac{x-3}{x^2-x-6} + \frac{2x-1}{2x^2+5x-3} - \frac{2x+5}{x^2+5x+6}$

Solution : (i) $\frac{x}{x+y} - \frac{y}{x-y} = \frac{x(x-y)-y(x+y)}{(x+y)(x-y)}$

$$= \frac{x^2-xy-xy-y^2}{x^2-y^2} = \frac{x^2-2xy-y^2}{x^2-y^2}$$

(ii) $\frac{1}{x^2-5x+6} + \frac{1}{x^2-3x+2} + \frac{2}{x^2-8x+15}$

$$= \left[\frac{1}{(x-3)(x-2)} + \frac{1}{(x-2)(x-1)}\right] + \left[\frac{2}{(x-5)(x-3)}\right]$$

$$= \left[\frac{x-1+x-3}{(x-1)(x-2)(x-3)}\right] + \left[\frac{2}{(x-5)(x-3)}\right]$$

$$= \frac{2x-4}{(x-1)(x-2)(x-3)} + \frac{2}{(x-5)(x-3)}$$

$$= \frac{2(x-2)}{(x-1)(x-2)(x-3)} + \frac{2}{(x-5)(x-3)}$$

$$= \frac{2}{(x-1)(x-3)} + \frac{2}{(x-5)(x-3)} = \frac{2(x-5)+2(x-1)}{(x-1)(x-3)(x-5)}$$

$$= \frac{4x-12}{(x-1)(x-3)(x-5)} = \frac{4}{x^2-6x+5}$$

(iii) $\frac{x-3}{x^2-x-6} + \frac{2x-1}{2x^2+5x-3} - \frac{2x+5}{x^2+5x+6}$

$$= \frac{x-3}{(x-3)(x+2)} + \frac{2x-1}{(2x-1)(x+3)} - \frac{2x+5}{(x+2)(x+3)}$$

$$= \left[\frac{1}{x+2}+\frac{1}{x+3}\right] - \left[\frac{2x+5}{(x+2)(x+3)}\right]$$

$$= \left[\frac{(x+3)+(x+2)}{(x+2)(x+3)}\right] - \left[\frac{2x+5}{(x+2)(x+3)}\right]$$

$$= \frac{2x+5}{(x+2)(x+3)} - \frac{2x+5}{(x+2)(x+3)} = 0$$

EXERCISE 5.12

I. Simplify :

1. $\frac{x^2+2}{x+1} + \frac{x^2-3}{x+1}$

2. $\frac{a^3}{a-b} + \frac{b^3}{b-a}$

3. $\frac{y-3}{y+2} - \frac{y-2}{y+3}$

4. $\frac{1}{x-1} - \frac{2}{x+1}$

5. If A = $\frac{2x+1}{2x-1}$, B = $\frac{2x-1}{2x+1}$ find A – B.

6. $\frac{x+1}{x-1} + \frac{x^2-1}{x+1}$

7. $\frac{1}{x^2-7x+12} + \frac{1}{x^2-5x+6}$

8. $\frac{x}{x-y} - \frac{y}{x+y} - \frac{2xy}{x^2-y^2}$

9. $\left[\frac{2x+5}{x+1}+\frac{x^2+1}{x^2-1}\right] - \left[\frac{3x-2}{x-1}\right]$

10. $\frac{1}{a^2+3a+2} + \frac{1}{a^2+5a+6} - \frac{2}{a^2+4a+3}$

5.5 UNDETERMINED COEFFICIENTS

Principle of undetermined coefficients.

I. Theorem : If two rational integral functions are *identical*, then the coefficients of the like powers of the variable can be equated.

Let P(x) be $p_n x^n + p_{n-1} x^{n-1} + p_{n-2} x^{n-2} + ...+ p_3 x^3 + p_2 x^2 + p_1 x + p_o$ and

Q(x) be $q_n x^n + q_{n-1} x^{n-1} + q_{n-2} x^{n-2} + ...+ q_3 x^3 + q_2 x^2 + q_1 x + q_o$

If P(x) = Q(x), then,

$p_n = q_n$; $p_{n-1} = q_{n-1}$; $p_{n-2} = q_{n-2}$; .... ; $p_3 = q_3$; $p_2 = q_2$; $p_1 = q_1$; $p_0 = q_0$

II. Corollary : The above principle still holds if one of the functions is of lower degree than the other.

Let R(x) = $r_n x^n + r_{n-1} x^{n-1} + r_{n-2} x^{n-2} + ...+ r_3 x^3 + r_2 x^2 + r_1 x + r_o$ and

S(x) = $s_{n-1} x^{n-1} + s_{n-2} x^{n-2} + ...+ s_3 x^3 + s_2 x^2 + s_1 x + s_o$

If R(x) = S(x), then, $r_n = 0$; $r_{n-1} = s_{n-1}$; $r_{n-2} = s_{n-2}$; .... ; $r_1 = s_1$; $r_0 = s_0$

Example 5.56 : Find m, n, p and q given
$3x^3 + 2x^2 + 4x + 1 \equiv mx^3 + nx^2 + px + q$.

Solution : $m = 3$; $n = 2$, $p = 4$, $q = 1$

Example 5.57 : If $x^2 - px + q \equiv (x - 3)^2$, find p and q.

Solution : $x^2 - px + q \equiv (x^2 - 6x + 9)$

$\therefore -p = -6, q = 9$

Hence, $p = 6$, $q = 9$.

Example 5.58 : If $x^2 - x - 6 \equiv (x + 2)(x - p)$, find '$p$'.

Solution : $x^2 - x - 6 \equiv x^2 + (2 - p)x - 2p$

$\therefore\ 2 - p = -1$ (or) $-2p = -6$

$\therefore\ p = 3$ $\quad p = 3$

Example 5.59 : If $(A + B)x + (A - B) \equiv 7x + 1$, find the values of A and B.

Solution :

$A + B = 7$ ... (1)

$A - B = 1$ ... (2)

(1) + (2) $\quad 2A = 8$

$\therefore\ A = 4$

$B = 3$

$\therefore\ A = 4, \quad B = 3$

Example 5.60 : Given $p(x - 1) + q(x - 3) \equiv 5x - 9$ find the values of p and q.

Solution : Set $x = 1$: $\quad -2q = -4$

$\therefore\ q = 2$

Set $x = 3$: $\quad 2p = 6$

$\therefore\ p = 3$

$\therefore\ p = 3,\ q = 2$

Example 5.61 : If $A(x - 1)(x - 2)(x - 3) + B(x - 1)(x - 2) + C(x - 1) + D \equiv 2x^3 - x - 3$, find the values of A, B, C and D.

Solution : Set $x = 1$ $\quad D = 2 - 1 - 3$

$\therefore\ D = -2$

Set $x = 2$: $\quad C + D = 16 - 2 - 3 = 11$

$C - 2 = 11$

$\therefore\ C = 13$

Set $x = 3$: $\quad 2B + 2C + D = 54 - 3 - 3 = 48$

$2B + 26 - 2 = 48$

$2B = 24$

$B = 12$

$$\text{Set } x = 0 : \quad -6A + 2B - C + D = -3$$

$$-6A + 24 - 13 - 2 = -3$$

$$-6A = -12$$

$$\therefore A = 2$$

Hence, A = 2, B = 12, C = 13, D = –2

EXERCISE 5.13

1. Find a, b, c and d given $ax^3 + bx^2 + cx + d \equiv 7x^3 - 10x^2 - 3x - 12$
2. If $(x + m)^2 \equiv x^2 + px + 9$, Find m and p.
3. Given $(2A + B)x + (A + B) \equiv 11x + 7$, find the values of A and B.
4. If $M(x + 3) + N(x - 2) \equiv 8x + 9$, find the values of M and N.
5. If $P(x + 1)(x + 2) + Q(x - 1) \equiv x^2 + 6x - 1$, find the values of P and Q.
6. Given $P(2 - x) + Q(1 - x)(1 + x) \equiv x - 2x^2$, find the values of P and Q.
7. If $A(x - 1)(x - 2)(x - 3) + B(x - 1)(x - 2) + C(x - 1) + D \equiv x^3$, find the values of A, B, C and D.
8. Given $A(x - 2)^3 + B(x - 2)^2 + C(x - 2) + D \equiv 2x^3 + 8x^2 + 22x + 22$ find the values of A, B, C and D.

5.6 PARTIAL FRACTIONS

In the applications of rational expression, it is often useful to decompose a rational expression into a sum of rational expressions with simpler denominators. We shall describe how this is done in certain cases.

By long division (if necessary), we can reduce any rational expression to the sum of a polynomial and the bottom - heavy rational expression.

For instance, $\dfrac{x^3 - 2x}{x^2 + x + 1} \equiv x - 1 + \dfrac{1 - 2x}{x^2 + x + 1}$

Therefore, we can concentrate on bottom - heavy rational expression.

5.6.1 The Partial Fractions Theorems

If P(x) and Q(x) are polynomials with the degree of P(x) less than the degree of Q(x), then $\dfrac{P(x)}{Q(x)}$ can be resolved into a sum of partial fractions as follows :

(1) If Q(x) has a linear factor ($ax + b$), then one term of the sum is the partial fraction $\frac{A}{ax+b}$, where A is a constant.

(2) If Q (x) has n repeated linear factors, $(ax + b)^n$, then n terms of the sum are the n partial fractions as follows :

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \ldots + \frac{A_n}{(ax+b)^n}$$

where $A_1, A_2, A_3, \ldots, A_n$ are constants.

(3) If Q(x) has a quadratic factor $ax^2 + bx + c$ (not factorisable), then one term of the sum is the partial fraction $\frac{Ax + B}{ax^2+bx+c}$, where A and B are constants.

Type I : Quadratic Denominator

We begin with rational expressions $\frac{ax+b}{(x-r)(x-s)}$ $(r \neq s)$

Such a rational expression can always be decomposed into two simple expressions as we shall see.

Example ***(1)*** $\frac{2}{(x-1)(x+1)} \equiv \frac{1}{x-1} + \frac{-1}{x+1}$

(2) $\frac{x}{(x+2)(x+1)} \equiv \frac{2}{x+2} + \frac{-1}{x+1}$

(3) $\frac{2x-3}{x(x+1)} \equiv \frac{-3}{x} + \frac{5}{x+1}$

In general, $\frac{ax+b}{(x-r)(x-s)} \equiv \frac{A}{x-r} + \frac{B}{x-s}$ for suitable constants A and B; $r \neq s$.

Example 5.62 : Decompose into partial fractions $\frac{3x+1}{(x+1)(x-2)}$

Solution : Write

$$\frac{3x+1}{(x+1)(x-2)} \equiv \frac{A}{x+1} + \frac{B}{x-2}$$ and clear of fractions:

$$3x + 1 \equiv A(x-2) + B(x+1)$$

Set $x = -1$, $-2 = -3A$ $\therefore$ A = 2/3

Set $x = 2$, $7 = 3B$ $\therefore$ B = 7/3

Hence, $\dfrac{3x+1}{(x+1)(x-2)} \equiv \dfrac{2}{3(x+1)} + \dfrac{7}{3(x-2)}$

Example 5.63 : Decompose into partial fractions.

$$\frac{2x^3-13x^2+22x-14}{(x-2)\,(x-3)}$$

Solution : Degree of the numerator is 3 . The degree of the denominator is 2. Hence, we have to divide actually.

$$(x-2)\,(x-3) = x^2 - 5x + 6$$

			2	–3		
1	–5	6	2	–13	22	–14
			2	–10	12	
				–3	10	–14
				–3	15	–18
					–5	4

$\therefore \dfrac{2x^3-13x^2+22x-14}{(x-2)\,(x-3)} \equiv (2x-3) + \dfrac{-5x+4}{(x-2)\,(x-3)}$

Now write, $\dfrac{-5x+4}{(x-2)\,(x-3)} \equiv \dfrac{A}{x-2} + \dfrac{B}{x-3}$ and clear of fractions :

$$-5x + 4 \equiv A\,(x-3) + B\,(x-2)$$

set $x = 2$, $-6 = -A$ $\therefore$ A = 6

set $x = 3$; $-11 = B$ $\therefore$ B = – 11.

$\therefore \dfrac{2x^3-13x^2+22x-14}{(x-2)\,(x-3)} \equiv 2x - 3 + \dfrac{6}{x-2} - \dfrac{11}{x-3}$

Type II : Now, we consider denominators of the form $(x-r)^2$. The typical such bottom - heavy rational expression $\dfrac{ax+b}{(x-r)^2}$ can be decomposed into $\dfrac{A}{x-r} + \dfrac{B}{(x-r)^2}$

$$\therefore \frac{ax+b}{(x-r)^2} \equiv \frac{A}{x-r} + \frac{B}{(x-r)^2}.$$

Example 5.64 : Decompose into partial fractions $\frac{2x^2-13x+22}{(x-4)^2}$.

Solution : First divide by $(x-4)^2$ $(= x^2 - 8x + 16)$

$$\begin{array}{ccc|ccc} & & & 2 & & \\ \hline 1 & -8 & 16 & 2 & -13 & 22 \\ & & & 2 & -16 & 32 \\ \hline & & & & 3 & -10 \\ \hline \end{array}$$

Therefore, $\frac{2x^2-13x+22}{(x-4)^2} = 2 + \frac{3x-10}{(x-4)^2}$

Now consider $\frac{3x-10}{(x-4)^2}$

$$\frac{3x-10}{(x-4)^2} \equiv \frac{A}{x-4} + \frac{B}{(x-4)^2}$$

$3x - 10 \equiv A\,(x-4) + B$

Set $x = 4$, B = 2 ;   Set $x = 0$, A = 3

$$\therefore \frac{2x^2-13x+22}{(x-4)^2} \equiv 2 + \frac{3}{x-4} + \frac{2}{(x-4)^2}$$

Example 5.65 : Decompose into partial fraction $\frac{3x-2}{(x-1)^2}$

Write $\frac{3x-2}{(x-1)^2} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2}$.

$3x - 2 \equiv A\,(x-1) + B$

Set $x = 1$, $B = 1$

Set $x = 0$: $-A + B = -2$ $\therefore$ $A = 3$

Hence, $\frac{3x-2}{(x-1)^2} \equiv \frac{3}{x-1} + \frac{1}{(x-1)^2}$

Type III Cubic Denoimintor

We shall survey the partial fraction decomposition of rational expression with cubic denominators. As usual, we can make a preliminary long division to assure a bottom heavy expression. We shall also always assume leading coefficients (in the denominator). Thus we are considering $\frac{P(x)}{Q(x)} = \frac{ax^2+bx+c}{Q(x)}$

where $Q(x) = x^3 + ...+ d$ (d - is constant)

Case (i) : $Q(x) = (x-r)(x-s)(x-t)$. The partial fraction decomposition is

$$\frac{A}{x-r} + \frac{B}{x-s} + \frac{C}{x-t} \quad (r \neq s \neq t)$$

Example 5.66 : Split into partial fractions $\frac{1}{(x-1)(x-2)(x-3)}$

Solution : Write $\frac{1}{(x-1)(x-2)(x-3)} \equiv \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$ clear of fractions :

$$A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) = 1$$

Substitute the special values $x = 1$, $x = 2$, $x = 3$, we get $A = \frac{1}{2}$, $B = -1$, $C = \frac{1}{2}$

$$\therefore \frac{1}{(x-1)(x-2)(x-3)} \equiv \frac{\frac{1}{2}}{x-1} + \frac{-1}{x-2} + \frac{\frac{1}{2}}{x-3}$$

Case (ii) : $Q(x) = (x-r)^2(x-s)$

The partial fraction decomposition is $\frac{A}{x-r} + \frac{B}{(x-r)^2} + \frac{C}{x-s}$

Example 5.67 (i) : Split into partial fractions $\frac{4x^2-x-9}{(x-1)^2(x+2)}$

Solution :

Write $\frac{4x^2-x-9}{(x-1)^2(x+2)} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$

Clear of fractions :

A $(x-1)(x+2)$ + B $(x+2)$ + C $(x-1)^2 \equiv 4x^2 - x - 9$

Set $x = 1$: 3B = –6 ∴ B = –2

Set $x = -2$: 9C = 9 ∴ C = 1

Set $x = 0$: –2A + 2B + C = – 9 ∴ A = 3

$$\therefore \frac{4x^2 - x - 9}{(x-1)^2(x+2)} \equiv \frac{3}{(x-1)} - \frac{2}{(x-1)^2} + \frac{1}{(x+2)}$$

Case (iii) : Q $(x) = (x - r)^3$

The partial fraction decomposition is $\frac{A}{x-r} + \frac{B}{(x-r)^2} + \frac{C}{(x-r)^3}$

Example 5.67 (ii) : Decompose into partial fractions $\frac{4x - x^2 + 3}{(x-1)^3}$

Solution : $\frac{4x - x^2 + 3}{(x-1)^3} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$

$\therefore 4x - x^2 + 3 \equiv A(x-1)^2 + B(x-1) + C$

Set $x = 1$, C = 6

Set $x = 0$, we get A – B = –3

Set $x = 2$, we get A + B = 1

Solving, we get A = –1, B = 2

$$\therefore \frac{-x^2 + 4x + 3}{(x-1)^3} = \frac{-1}{x-1} + \frac{2}{(x-1)^2} + \frac{6}{(x-1)^3}$$

Case (iv) : Q $(x) = (x - r)(x^2 + b\,x + c)$, where the quadratic factor has no factors. The partial decomposition is $\frac{A}{x-r} + \frac{Bx + C}{x^2 + bx + c}$

Example 5.68 : Decompose into partial fractions $\frac{x^2 - 5x - 2}{(x-3)(x^2+1)}$

Solution : Write $\frac{A}{x-3} + \frac{Bx + C}{x^2+1} \equiv \frac{x^2 - 5x - 2}{(x-3)(x^2+1)}$

Clear of fractions :

$A(x^2+1)+(Bx+C)(x-3) \equiv x^2-5x-2$

Set $x = 3$, $10A = -8$ $\qquad \therefore A = \frac{-8}{10} = \frac{-4}{5}$

We could now substitute $x = 0$ to determine C and then substitute, say $x = 1$ to determine B.

$$\therefore B = \frac{9}{5} \text{ and } C = \frac{2}{5}$$

$$\therefore \frac{x^2-5x-2}{(x-3)(x^2+1)} = \frac{\frac{-4}{5}}{x-3} + \frac{\frac{9}{5}x+\frac{2}{5}}{x^2+1}$$

EXERCISE 5.14

Decompose into partial fractions :

1. $\frac{2}{x(x+4)}$ 2. $\frac{x}{x^2-1}$, 3. $\frac{-x}{(x+1)(x+2)}$ 4. $\frac{x+1}{x(x-1)}$

5. $\frac{2x-3}{(x+2)(x+3)}$ 6. $\frac{x^2}{(2x+1)(2x-1)}$ 7. $\frac{x}{x^2+2x+1}$

8. $\frac{x}{(x+1)(x+2)(x+3)}$ 9. $\frac{x-1}{(3x+2)(x+3)(x+4)}$ 10. $\frac{x^2+x+1}{(x-2)^2(x+2)}$

11. $\frac{3x^2-1}{(x-2)^3}$ 12. $\frac{x^2}{(x+3)(x^2+1)}$

5.7 COMPUTATION OF SQUARE ROOT

The square root of a given number is another number which when multiplied with itself results in the given number.

Similarly, the square root of a given polynomial $P(x)$ is another polynomial $Q(x)$ which when multiplied by itself gives $P(x)$.

In the earlier classes you learnt to find square root of polynomials by factorisation method. Here, we find square root by the method of division.

For polynomials of higher degree the method of division is very much useful. This method is similar to the division method of finding the square root of numbers.

We consider the first digit and take its square root as a starting step. At each successive step, the next two terms in the given polynomial are to be brought down for the division process, as in the case of numbers.

The following example explain the method:

Example 5.69 : Find the square root of $4x^4 + 12x^3 + x^2 - 12x + 4$

Steps for calculating square root of a polynomial by the method of division:

1. Arrange the given polynomial in descending powers of x.
2. Draw lines above and to the side of the polynomial as shown. The quotient is written above the horizontal line and the divisor for each stage is written to the left of the vertical line.

$$
\begin{array}{l|l}
 & \quad 2x^2 + 3x - 2 \\
\hline
2x^2 & 4x^4 + 12x^3 + x^2 - 12x + 4 \\
 & 4x^4 \\
\hline
4x^2 + 3x & \qquad 12x^3 + x^2 \\
 & \qquad 12x^3 + 9x^2 \\
\cline{2-2}
4x^2 + 6x - 2 & \qquad\qquad -8x^2 - 12x + 4 \\
 & \qquad\qquad -8x^2 - 12x + 4 \\
\cline{2-2}
 & \qquad\qquad\qquad 0
\end{array}
$$

3. The square root of the first term $4x^4$ is $2x^2$, written as the first term of the quotient above the horizontal line and also as the divisor to the left of the vertical line.
4. The product of $2x^2$ above the horizontal line and $2x^2$ on the left of the vertical line is $4x^4$ is written below the first term $4x^4$ and subtracted.
5. Bring down the next two terms of the given polynomial to form the new dividend $12x^3 + x^2$.
6. Multiply the quotient $2x^2$ by 2 and write the product $4x^2$ to the left side of the vertical line as the new divisor.
7. Divide the first term of the new dividend by the new divisor which is $\frac{12x^3}{4x^2} = 3x$. Now $3x$ is the second term of the quotient and written to the right of $2x^2$ above the horizontal line. Also $3x$ is written to the right of $4x^2$ as shown. Now $4x^2 + 3x$ is the new divisor.

8. Multiply $4x^2 + 3x$ by $3x$ and we get $12x^3 + 9x^2$ which is written below the new dividend $12x^3 + x^2$ and subtracted yielding the remainder $-8x^2$.

9. Bring down the last two terms to get the new dividend $-8x^2 - 12x + 4$.

10. Multiply the current quotient by 2 and we get $4x^2 + 6x$ which is written to the left of the vertical line as a part of the new divisor.

11. Divide the first term of the new dividend by the first term of the new divisor $\frac{-8x^2}{4x^2} = -2$ which is the last term of the quotient and is written to the right of $2x^2 + 3x$ above the horizontal line and also to the right of the new divisor $4x^2 + 6x$.

12. Multiply $4x^2 + 6x - 2$ by -2 and write below the new dividend and subtract.

13. The remainder is 0. There are no more terms left in the given polynomial and the quotient $2x^2 + 3x - 2$ is the square root of the given polynomial.

Hence, $\sqrt{4x^4 + 12x^3 + x^2 - 12x + 4} = 2x^2 + 3x - 2$

Example 5.70 : Find the square root of $4x^2 + 12xy + 9y^2 + 16x + 24y + 16$

Solution :

	$2x + 3y + 4$
$2x$	$4x^2 + 12xy + 9y^2 + 16x + 24y + 16$
	$4x^2$
$4x + 3y$	$12xy + 9y^2$
	$12xy + 9y^2$
$4x + 6y + 4$	$16x + 24y + 16$
	$16x + 24y + 16$
	0

Hence, $\sqrt{4x^4 + 12xy + 9y^2 + 16x + 24y + 16} = 2x + 3y + 4$

Example 5.71 : Find the square root of $4\frac{x^2}{y^2} + 20\frac{x}{y} + 13 - 30\frac{y}{x} + 9\frac{y^2}{x^2}$

Solution:

$$
\begin{array}{r|l}
 & 2\frac{x}{y} + 5 - \frac{3y}{x} \\
\hline
2\frac{x}{y} & 4\frac{x^2}{y^2} + 20\frac{x}{y} + 13 - 30\frac{y}{x} + 9\frac{y^2}{x^2} \\
 & 4\frac{x^2}{y^2} \\
\hline
4\frac{x}{y} + 5 & \qquad 20\frac{x}{y} + 13 \\
 & \qquad 20\frac{x}{y} + 25 \\
\hline
4\frac{x}{y} + 10 - \frac{3y}{x} & \qquad\qquad -12 - 30\frac{y}{x} + 9\frac{y^2}{x^2} \\
 & \qquad\qquad -12 - 30\frac{y}{x} + 9\frac{y^2}{x^2} \\
\hline
 & \qquad\qquad\qquad 0
\end{array}
$$

Hence $\sqrt{\frac{4x^2}{y^2} + 20\frac{x}{y} + 13 - 30\frac{y}{x} + \frac{9y^2}{x^2}} = \frac{2x}{y} + 5 - \frac{3y}{x}$

Example : 5.71 If $9x^4 + 12x^3 + 28x^2 + ax + b$ is a perfect square, find the values of a and b.

Solution :

$$
\begin{array}{r|l}
 & 3x^2 + 2x + 4 \\
\hline
3x^2 & 9x^4 + 12x^3 + 28x^2 + ax + b \\
 & 9x^4 \\
\hline
6x^2 + 2x & \quad 12x^3 + 28x^2 \\
 & \quad 12x^3 + 4x^2 \\
\hline
6x^2 + 4x + 4 & \qquad 24x^2 + ax + b \\
 & \qquad 24x^2 + 16x + 16 \\
\hline
 & \qquad\qquad 0
\end{array}
$$

Because the given polynomial is a perfect square.

$\therefore\ a = 16$ and $b = 16$.

Another method :

Detatched coefficient Method

Example 5.72 : Find the square root of $x^4 - 10x^3 + 37x^2 - 60x + 36$

Method : Arrange the given polynomial in descending power of x. Write the coefficients of x^4, x^3, x^2, x and constant in a line. Proceed with the same steps as explained in example 1.

Solution :

			1	–5	6		
		1	1	–10	37	–60	36
			1				
	2	–5		–10	37		
				–10	25		
2	–10	6			12	–60	36
					12	–60	36
						0	

As the degree of the given polynomial is 4, the degree of its square root must be 2.

Hence, $\sqrt{x^4 - 10x^3 + 37x^2 - 60x + 36} = x^2 - 5x + 6$ (written using the numbers 1, –5, 6 above the horizontal line).

Example 5.73 : Find the square root of $\frac{x^2}{y^2} - 10\frac{x}{y} + 27 - 10\frac{y}{x} + \frac{y^2}{x^2}$

Solution :

			1	–5	1		
		1	1	–10	27	–10	1
			1				
	2	–5		–10	27		
				–10	25		
2	–10	1			2	–10	1
					2	–10	1
						0	

Hence $\sqrt{\frac{x^2}{y^2} - 10\frac{x}{y} + 27 - 10\frac{y}{x} + \frac{y^2}{x^2}} = 1\left(\frac{x}{y}\right) - 5 + 1\left(\frac{y}{x}\right)$

$$= \frac{x}{y} - 5 + \frac{y}{x}$$

EXERCISE 5. 15

I. Find the square root of the following :

(a) $x^4 + 10x^3 + 31x^2 + 30x + 9$ (b) $9x^4 - 6x^3 + 7x^2 - 2x + 1$

(c) $4 + 25a^2 - 12a - 24a^3 + 16a^4$ (d) $16x^4 - 24x^3y + 49x^2y^2 - 30xy^3 + 25y^4$

(e) $4\frac{x^2}{y^2} + 8\frac{x}{y} + 16 + 12\frac{y}{x} + 9\frac{y^2}{x^2}$ (f) $\frac{x^2}{y^2} + 2\frac{x}{y} + \frac{y^2}{x^2} + 3 + \frac{2y}{x}$

II. Find the value of a for the following polynomials to be perfect squares.

(a) $4x^4 - 12x^3 + 37x^2 - 42x + a$ (b) $x^4 - 4x^3 + 10x^2 - ax + 9$

III. Find the values of a and b if the following polynomials are to be perfect squares.

(a) $x^4 + 4x^3 + 16x^2 + ax + b$ (b) $49x^4 - 70x^3 + 109x^2 + ax - b$

(c) $x^4 - 2x^3 - \left(\frac{3}{2}\right)x^2 + ax + b$ (d) $\frac{1}{x^4} - \frac{6}{x^3} + \frac{13}{x^2} + \frac{a}{x} + b$

5.8 QUADRATIC EQUATION

An equation of the form $ax^2 + bx + c = 0$, where a, b, c are real numbers with $a \neq 0$ is called a quadratic equation in the variable x.

A value of x which satisfies a given quadratic equation is called its root (or solution). Every quadratic equation has two and only two roots, may be distinct or equal. By solving a quadratic equation, we mean finding its roots and forming the solution set containing them.

5.8.1 Solving Quadratic Equation by Factorisation Method.

In this section, the method of factorisation is applied to solve quadratic equation. This method can be used when the quadratic equation can be factorised into two linear factors.

Example 5.74 : Solve the quadratic equation $2x^2 + 5x - 25 = 0$

Solution : $2x^2 + 5x - 25 = 0$

$\Rightarrow \quad 2x^2 + 10x - 5x - 25 = 0$

$\Rightarrow \quad 2x(x + 5) - 5(x + 5) = 0$

$\Rightarrow \quad (2x - 5)(x + 5) = 0$

$\Rightarrow \quad 2x - 5 = 0 \quad \text{or} \quad x + 5 = 0$

$\Rightarrow \quad x = \frac{5}{2} \quad \text{or} \quad x = -5$

Hence the roots of the given equation are $-5, \frac{5}{2}$

[**Note:** If $a \times b = 0$ either $a = 0$ or $b = 0$]

Example 5.75 : Solve $2x^2 = 3x$

Solution : $2x^2 = 3x$

$\Rightarrow \quad 2x^2 - 3x = 0$

$\Rightarrow \quad x(2x - 3) = 0$

$\Rightarrow \quad x = 0 \quad \text{or} \quad 2x - 3 = 0$

$\Rightarrow \quad x = 0 \quad \text{or} \quad x = \frac{3}{2}$

$\therefore$ Solution set is $\left\{0, \frac{3}{2}\right\}$

Example 5.76 : Solve $\sqrt{3}\,x^2 + 11x + 6\sqrt{3} = 0$

Solution : $\sqrt{3}\,x^2 + 11x + 6\sqrt{3} = 0$

$\Rightarrow \quad \sqrt{3}\,x^2 + 9x + 2x + 6\sqrt{3} = 0$

$\Rightarrow \sqrt{3}\,x(x + 3\sqrt{3}) + 2(x + 3\sqrt{3}) = 0$

$\Rightarrow \quad (x + 3\sqrt{3})(\sqrt{3}\,x + 2) = 0$

$\Rightarrow \quad x = -3\sqrt{3}$ or $x = \dfrac{-2}{\sqrt{3}}$

$\therefore$ Solution set is $\left\{-3\sqrt{3},\ \dfrac{-2}{\sqrt{3}}\right\}$

Example 5.77 : Solve $\dfrac{2x}{x-3} + \dfrac{1}{2x+3} + \dfrac{3x+9}{(x-3)(2x+3)} = 0$

Solution : Obviously, the given equation is valid, if $x - 3 \neq 0$ and $2x + 3 \neq 0$.

Multiplying throughout by $(x - 3)(2x + 3)$, we get

$$2x(2x+3) + 1(x-3) + 3x + 9 = 0$$
$$4x^2 + 10x + 6 = 0$$
$$\text{(i.e.,) } 2x^2 + 5x + 3 = 0$$
$$2x^2 + 3x + 2x + 3 = 0$$
$$x(2x+3) + 1(2x+3) = 0$$
$$(2x+3)(x+1) = 0$$

But $\quad 2x + 3 \neq 0$

So $\quad x + 1 = 0$

$\Rightarrow x = -1$

$\therefore$ the solution set is $\{-1\}$

Example 5.78 : Solve $\sqrt{2x+9} = 13 - x$

Solution : $\sqrt{2x+9} = 13 - x$

Squaring both sides, we get

$$2x + 9 = (13 - x)^2$$
$$2x + 9 = 169 - 26x + x^2$$
$$2x + 9 - 169 + 26x - x^2 = 0$$
$$-x^2 + 28x - 160 = 0$$
$$x^2 - 28x + 160 = 0$$
$$(x - 8)(x - 20) = 0$$

Remark : Clearly we have to find those solutions for which $2x + 9 \geq 0$ and $13 - x \geq 0$. Since $x = 20$ does not satisfy these conditions, it is an extraneous root. $\therefore$ $x = 8$ is the only root of the given equation.

$$x - 8 = 0 \quad \text{or } x - 20 = 0$$

$$x = 8 \quad \text{or} \quad x = 20$$

But $x = 20$, does not satisfy the given equation, so $x = 20$ is rejected. Hence the solution set is { 8 }.

Note : When squaring on both sides of the equation is done, the roots of the final equation must be checked to determine whether they are roots of the original equation or not.

Although no root of the original equation will be lost by squaring but certain values may be introduced which are roots of the new equation but not of the original equation.

EXERCISE 5.16

Solve the following quadratic equation by factorisation method.

1. $9x^2 - 16 = 0$
2. $(2x + 3)(3x - 7) = 0$
3. $(x - 2)^2 - 25 = 0$
4. $(2y + 3)^2 = 81$
5. $y^2 - 5 = 0$
6. $a^2 z^2 - b^2 = 0$
7. $4y^2 + 4y + 1 = 0$
8. $3x^2 - 5x - 12 = 0$
9. $4\sqrt{5}\,x^2 + 7x - 3\sqrt{5} = 0$
10. $\sqrt{2}\,x^2 - 3x - 2\sqrt{2} = 0$
11. $3(y^2 - 6) = y(y + 7) - 3$
12. $3x - \dfrac{8}{x} = 2$
13. $\dfrac{x}{x-1} + \dfrac{x-1}{x} = 2\,\dfrac{1}{2}$
14. $\sqrt{3x+4} = x$
15. $\sqrt{x(x-7)} = 3\sqrt{2}$
16. $a^2 b^2 x^2 - (a^2 + b^2)x + 1 = 0$
17. $\left(x - \dfrac{a}{b}\right)^2 = \dfrac{a^2}{b^2}$

5.8.2 Solving Quadratic Equations by the method of completion of square

Every quadratic equation can be solved by completing the squares. The examples below illustrate this process.

Example 5.79 : Solve $x^2 + 3x + 1 = 0$

Solution : $x^2 + 3x + 1 = 0$

We add and substract ($\frac{1}{2}$ coefficient of x)^2 in LHS and get

$$x^2 + 3x + 1 + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 = 0$$

$$\Rightarrow \quad x^2 + 2\left(\frac{3}{2}\right)x + 1 + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 = 0$$

$$\Rightarrow \quad x^2 + 2\left(\frac{3}{2}\right)x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 1 = 0$$

$$\Rightarrow \quad \left(x+\frac{3}{2}\right)^2 - \frac{9}{4} + 1 = 0$$

$$\Rightarrow \quad \left(x+\frac{3}{2}\right)^2 = \frac{5}{4}$$

$$\Rightarrow \quad x + \frac{3}{2} = \pm\frac{\sqrt{5}}{2}$$

$$\Rightarrow \quad x = \frac{-3 \pm \sqrt{5}}{2}$$

Example 5.80 : Solve, Using the completion of square method.

$2x^2 + 5x - 3 = 0$

Solution : $2x^2 + 5x - 3 = 0$.

$$\Rightarrow \quad x^2 + \frac{5}{2}x - \frac{3}{2} = 0$$

$$\Rightarrow \quad \left(x+\frac{5}{4}\right)^2 - \left(\frac{5}{4}\right)^2 - \frac{3}{2} = 0$$

$$\Rightarrow \quad \left(x+\frac{5}{4}\right)^2 = \frac{49}{16}$$

$$x + \frac{5}{4} = \pm\frac{7}{4}$$

$$x = \frac{1}{2} \quad \text{or } -3.$$

5.8.3 Solution of quadratic equation by formula method

Some problems are easily solved by using a formula. Such in the case for solving quadratic equations. There is a formula that allow you to quickly write down the solution of any quadratic equation called the quadratic formula which is developed below.

Consider the general quadratic equation $ax^2 + bx + c = 0$, where a, b, c are real and $a \neq 0$. To find the formula, solve the equation by completing the square. We have, $ax^2 + bx + c = 0$

Dividing throughout by a, we get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{-c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \text{[completing the square method]}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-c}{a} + \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore\ x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Hence the quadratic equation $ax^2 + bx + c = 0$ has solution $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Example 5.81 : Solve using the quadratic formula

$$x^2 - 4x + 1 = 0$$

Solution : $x^2 - 4x + 1 = 0$

Comparing it with $ax^2 + bx = c = 0$, we get $a = 1$, $b = -4$, $c = 1$.

By using the formula , $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ we get

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4.1.1}}{2.1}$$

$$= \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

Hence the roots of the given equation are $2 + \sqrt{3}$, $2 - \sqrt{3}$

Example 5.82 : Solve $\dfrac{1}{x+1} + \dfrac{2}{x+2} = \dfrac{4}{x+4}$

Solution : $\dfrac{1}{x+1} + \dfrac{2}{x+2} = \dfrac{4}{x+4}$

$$\frac{1(x+2)+2(x+1)}{(x+1)(x+2)} = \frac{4}{x+4} \Rightarrow \frac{3x+4}{(x+1)(x+2)} = \frac{4}{x+4}$$

$\Rightarrow$ $(3x + 4)(x + 4) = 4(x + 1)(x + 2)$

$\Rightarrow$ $3x^2 + 12x + 4x + 16 = 4(x^2 + 2x + x + 2)$

$\Rightarrow$ $3x^2 + 16x + 16 = 4x^2 + 12x + 8$

$\Rightarrow$ $x^2 - 4x - 8 = 0$

Comparing it with $ax^2 + bx + c = 0$, we get $a = 1$, $b = -4$, $c = -8$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{(quadratic formula)}$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4.1(-8)}}{2.1}$$

$$= \frac{4 \pm \sqrt{16+32}}{2} = \frac{4 \pm \sqrt{48}}{2} = \frac{4 \pm 4\sqrt{3}}{2}$$

$$\therefore \quad x = 2 \pm 2\sqrt{3}$$

Hence the roots are $2 + 2\sqrt{3}$, $2 - 2\sqrt{3}$.

Example 5.83 : Solve $abx^2 - (a+b)^2x + (a+b)^2 = 0$

Solution : $abx^2 - (a+b)^2x + (a+b)^2 = 0$

Comparing it with $Ax^2 + Bx + C = 0$

We have $A = ab$, $B = -(a+b)^2$, $C = (a+b)^2$

$$\therefore x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$= \frac{(a+b)^2 \pm \sqrt{(a+b)^4 - 4ab(a+b)^2}}{2ab}$$

$$= \frac{(a+b)^2 \pm \sqrt{(a+b)^2\,[(a+b)^2 - 4ab]}}{2ab}$$

$$= \frac{(a+b)^2 \pm \sqrt{(a+b)^2\,(a-b)^2}}{2ab}$$

$$x = \frac{(a+b)^2 \pm (a+b)\,(a-b)}{2ab}$$

$$x = \frac{(a+b)^2 \pm (a^2 - b^2)}{2ab}$$

$$x = \frac{(a+b)^2 + (a^2 - b^2)}{2ab} \quad \text{or} \quad x = \frac{(a+b)^2 - (a^2 - b^2)}{2ab}$$

$$= \frac{a^2 + b^2 + 2ab + a^2 - b^2}{2ab} \quad \text{or } x = \frac{a^2 + b^2 + 2ab - a^2 + b^2}{2ab}$$

$$= \frac{2a^2 + 2ab}{2ab} \qquad \text{or} \quad x = \frac{2ab + 2b^2}{2ab}$$

$$= \frac{2a(a+b)}{2ab} \qquad \text{or} \quad x = \frac{2b(a+b)}{2ab}$$

$$= \frac{a+b}{b} \qquad \text{or} \quad x = \frac{a+b}{a}$$

Hence the solution is $\left\{ \frac{a+b}{b}, \frac{a+b}{a} \right\}$

EXERCISE- 5.17

I. Solve the following quadratic equation using completion of square method.

1. $x^2 + 10x + 9 = 0$
2. $x^2 - 4x - 45 = 0$
3. $5x^2 + 14x = 55$
4. $15 = 17x + 4x^2$
5. $\frac{5x+7}{x-1} = 3x + 2$

II. Solve the following quadratic equation using quadratic formula :

1. $x^2 + 2x - 2 = 0$
2. $x^2 - 6x - 3 = 0$
3. $2x^2 - 3x - 5 = 0$
4. $4x^2 + 7x + 2 = 0$
5. $(x - 3)^2 = 2(x + 4)$
6. $3x^2 + 2\sqrt{5}\,x - 5 = 0$
7. $\sqrt{x+5} = 2x + 3$
8. $a(x^2 + 1) = x(a^2 + 1)$
9. $3a^2x^2 - abx - 2b^2 = 0$
10. $4x^2 - 2(a^2 + b^2)x + a^2b^2 = 0$
11. $4x^2 - 4a^2x + (a^4 - b^4) = 0$
12. $p^2x^2 + (p^2 - q^2)x - q^2 = 0$
13. $36x^2 - 12ax + (a^2 - b^2) = 0$

5.8.4 Problems leading to quadratic equations

In this section, we will discuss some simple problems on practical applications of quadratic equation. In this we have to form the equation based on the given condition and then to solve the equation.

Example 5.84 : The sum of the squares of two consecutive natural numbers is 313. Find the numbers.

Solution : Let the two consecutive natural numbers be x and $x + 1$,

Then, $x^2 + (x + 1)^2 = 313$

$\Rightarrow \quad 2x^2 + 2x - 312 = 0$

$\Rightarrow \quad x^2 + x - 156 = 0$

$\Rightarrow \quad (x + 13)(x - 12) = 0$

$\Rightarrow \quad x + 13 = 0 \quad$ or $x - 12 = 0$

$\Rightarrow \quad x = -13 \quad$ or $\quad x = 12$

Since x, being a natural number, cannot be negative $\therefore x = 12$. Hence the two consecutive natural numbers are 12 and 13.

Example 5.85 : The sum of two natural numbers is 8. Determine the numbers, if the sum of their reciprocals is $\frac{8}{15}$.

Solution : As the sum of two natural numbers is 8, let the numbers be x and $8 - x$. Then

$$\Rightarrow \quad \frac{1}{x} + \frac{1}{8-x} = \frac{8}{15}$$

$$\Rightarrow \quad \frac{8}{x(8-x)} = \frac{8}{15}$$

$$\Rightarrow \quad x(8-x) = 15$$

$$\Rightarrow \quad x^2 - 8x + 15 = 0$$

$$\Rightarrow \quad x = 3 \text{ or } x = 5$$

When $x = 3$, the numbers are 3 and 8 – 3 (i.e.,) 3 and 5

When $x = 5$, the numbers are 5 and 8 – 5 (i.e.,) 5 and 3.

Hence the required numbers are 3 , 5.

Example 5.86 : The denominator of a fraction is one more than twice the numerator. If the sum of the fraction and its reciprocal is $2\frac{16}{21}$, find the fraction.

Solution : Let the numerator of the fraction be x, then its denominator is $2x+1$. So the fraction is $\frac{x}{2x+1}$.

$$\text{Then} \qquad \frac{x}{2x+1} + \frac{2x+1}{x} = 2\frac{16}{21}$$

$$\Rightarrow \qquad \frac{x^2+(2x+1)^2}{(2x+1)x} = \frac{58}{21}$$

$$\Rightarrow \qquad 58x\,(2x + 1) = 21\,(x^2 + 4x^2 + 4x + 1)$$

$$\Rightarrow \qquad 116x^2 + 58x = 105x^2 + 84x + 21$$

$$\Rightarrow \qquad 11x^2 - 26x - 21 = 0$$

$$\Rightarrow \qquad x = 3 \quad \text{or } x = \frac{-7}{11}$$

$\Rightarrow x = 3$ [since x is a natural number, $\therefore\ x > 0$]

$\therefore$ Required fraction = $\frac{x}{2x+1} = \frac{3}{2(3)+1} = \frac{3}{7}$

Example 5.87 : A two digit number is such that the product of its digits is 12. When 36 is added to this number, the digits are interchanged. Find the number.

Solution : Let the units digit of the two digit number be x.

Since the product of its digits is 12, its tens' digit is $\frac{12}{x}$

$\therefore$ The number is $10\left(\frac{12}{x}\right) + x$.

On interchanging the digits, the number = $10\,(x) + \frac{12}{x}$

$$\text{Given,} \qquad 10\left(\frac{12}{x}\right) + x + 36 = 10\,(x) + \frac{12}{x}$$

$$\Rightarrow \qquad \frac{120}{x} + x + 36 = 10x + \frac{12}{x}$$

$$\Rightarrow \quad 120 + x^2 + 36x = 10x^2 + 12$$

$$\Rightarrow \quad x^2 - 4x - 12 = 0$$

$$\Rightarrow \quad x - 6 = 0 \text{ or } x + 2 = 0$$

$x = 6$ or $x = -2$, but x being a digit of a number cannot be negative

$$\therefore x = 6$$

$\therefore$ Unit's digit = 6 and ten's digit = $\dfrac{12}{x} = \dfrac{12}{6} = 2$

Hence the required number is 26.

Example 5.88 : The hypotenuse of a right angled triangle is 17cm and the difference between other two sides is 7 cm. Find the other two unknown sides.

Solution : Let the one side be x cm. since the difference between the two sides is 7 cm,

$$\therefore \text{ other side } = (x + 7) \text{ cm.}$$

As the given triangle is a right angled triangle with hypotenuse = 17cm, by using Pythagoras theorem, we get

$$x^2 + (x + 7)^2 = (17)^2$$

$$\Rightarrow \quad x^2 + 7x - 120 = 0$$

$$\Rightarrow \quad (x + 15)(x - 8) = 0$$

$$\Rightarrow \quad x + 15 = 0 \text{ or } x - 8 = 0$$

$$x = -15 \text{ or } x = 8 \text{ but } x \text{ cannot be negative.}$$

$$\therefore x = 8$$

Hence the two sides of the triangle are 8 cm and 15 cm.

Example 5.89 : A train covers a distance of 90km at a uniform speed. Had the speed been 15 km/hr more, it would have taken 30 minutes less for the journey. Find the original speed of the train.

Solution : Let the original speed of the train be x km/hr,

$\therefore$ time taken to cover a distance of 90 km. = $\dfrac{90}{x}$ hrs

The new speed of the train = $(x + 15)$ km / hr

$\therefore$ new time taken to cover 90 km = $\frac{90}{x+15}$ hrs.

Since the train takes 30 minutes (i.e.,) hour less,

$$\frac{90}{x} - \frac{90}{x+15} = \frac{1}{2} \qquad \text{[Since 30 min} = \frac{1}{2}\text{ hr.]}$$

$$\Rightarrow \quad \frac{90(x+15)-90x}{x(x+15)} = \frac{1}{2}$$

$$\Rightarrow \quad 2\,[90x + 1350 - 90x] = x^2 + 15x$$

$$\Rightarrow \quad 2700 - x^2 - 15x = 0$$

$$\Rightarrow \quad x^2 + 15x - 2700 = 0$$

$$\Rightarrow \quad (x + 60)(x - 45) = 0$$

$$\therefore x = -60 \text{ or } x = 45 \text{ but } x \text{ cannot be negative}$$

$$\therefore x = 45$$

Hence the original speed of the train = 45 km / hr.

EXERCISE 5.18

1. Find two consecutive positive odd numbers, the sum of whose squares is 802.
2. The sum of a number and its reciprocal is $2\frac{1}{30}$ Find the number.
3. The sum S of first n natural numbers is given by the formula S = $\frac{n(n+1)}{2}$ If S = 231, find n.
4. The difference of the squares of two positive numbers is 45. The square of the smaller number is four times the larger number. Find the numbers.
5. The product of two successive multiples of 4 is 28 more than the first multiple of 4. Find the multiples.
6. Divide 16 into two parts such that twice the square of the larger part exceeds the square of the smaller part by 164.
7. A two digit number contains the smaller of two digits in the units' place. The product of the digits is 24 and the difference between the digits is 5. Find the number.

8. A two digit number is such that the product of the digits is 18. If 27 is subtracted from the number the digits are interchanged. Find the number.
9. The sum of the ages of a father and his son is 45 years. Five years ago, the product of their ages (in years) was 124. Determine their present ages.
10. The sides of a right angled triangle are $2x - 1$ $2x$, and $2x + 1$, Find x.
11. The perimeter of a rectangle is 36 cm and its area is 80 sq. cm. Find its dimensions.
12. An aeroplane left 30 minutes later than its scheduled time, and in order to reach its destination 1500 km away in time, it has to increase its speed by 250 km / hr from its usual speed. Determine its usual speed.

5.8.5 Nature of Roots

The roots of the quadratic equation $ax^2 + bx + c = 0$ are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

The nature of the roots depends on the value of $b^2 - 4ac$. The value of the expression $b^2 - 4ac$ discriminant the nature of roots and so it is called the discriminant of the quadratic equation. It is denoted by the symbol Δ.

	Discriminant $\Delta = b^2 - 4ac$	**Nature of roots**
1.	$\Delta > 0$ but not a perfect square.	Real, unequal and irrational.
2.	$\Delta > 0$ and a perfect square.	Real, unequal and rational.
3.	$\Delta = 0$	Real and equal
4.	$\Delta < 0$	Unreal.

Example 5.90 : Determine the nature of the roots of the following quadratic equations : (a) $2x^2 + x - 1 = 0$; (b) $x^2 - 4x + 4 = 0$; (c) $2x^2 + 5x + 5 = 0$

Solution : (a) $2x^2 + x - 1 = 0$

Here $a = 2$, $b = 1$, $c = -1$

$$\therefore \Delta = b^2 - 4ac$$

$$= 1^2 - (4)(2)(-1) = 1 + 8 = 9$$

Since $\Delta > 0$ and a perfect square, the roots of the given equation are real, distinct and rational.

(b) $x^2 - 4x + 4 = 0$

Here $a = 1$, $b = -4$, $c = 4$

$$\Delta = b^2 - 4ac = (-4)^2 - 4(1)(4) = 16 - 16 = 0$$

Since $\Delta = 0$, the roots of the given equation are real and equal.

(c) $2x^2 + 5x + 5 = 0$

Here $a = 2$, $b = 5$, $c = 5$

$$\Delta = b^2 - 4\,ac = 5^2 - 4\,(2)\,(5) = 25 - 40 = -15$$

Since $\Delta < 0$, the roots of the given equation are unreal.

Example 5.91 : If p, q, r are real and $p \neq q$, then show that the roots of the equation $(p - q)x^2 + 5\,(p + q)\,x - 2\,(\,p - q) = 0$ are real and unequal.

Solution : $(p - q)\,x^2 + 5\,(p + q)\,x - 2\,(p - q) = 0$

Here $a = p - q$, $b = 5\,(p + q)$, $c = -2\,(p - q)$

$$\therefore\ \Delta = b^2 - 4ac = 25\,(\,p + q)^2 - 4\,(p - q)\,(-2)\,(p - q)$$

$$= 25\,(p + q\,)^2 + 8\,(p - q\,)^2$$

It is clear that $25\,(p + q)^2 > 0$ and $8(p - q)^2 > 0$

$$\therefore\ \Delta = 25\,(p + q)^2 + 8\,(p - q)^2 > 0$$

Hence the roots of the given equation are real and unequal.

Example 5.92 : Show that the roots of the equation

$x^2 + 2(a + b)x + 2(a^2 + b^2) = 0$ are unreal.

Solution : $x^2 + 2(a + b)\,x + 2\,(a^2 + b^2) = 0$

Here $A = 1$, $B = 2\,(a + b)$, $C = 2\,(a^2 + b^2)$

$$\therefore\ \Delta = B^2 - 4AC = [2\,(a + b)]^2 - 4\,(1)(2)\,(a^2 + b^2)$$

$$= 4\,(a^2 + 2ab + b^2) - 8a^2 - 8b^2$$

$$= -4a^2 + 8ab - 4b^2$$

$$= -4\,(a^2 - 2ab + b^2)$$

$$= -4\,(a - b)^2$$

Since $(a - b)^2$ is positive , $-4(a - b)^2$ is negative.

Hence the roots are unreal.

Example 5.93 : Find the value of k for which the given equation $9x^2 + 3kx+4=0$ has real and equal roots

Solution : $9x^2 + 3kx + 4 = 0$

Here $a = 9$, $b = 3k$, $c = 4$

Given : The equation has real and equal roots

$$\Rightarrow \quad \Delta = 0$$
$$\Rightarrow \quad b^2 - 4\,ac = 0$$
$$\Rightarrow \quad (3k)^2 - 4(9)(4) = 0$$
$$\Rightarrow \quad 9k^2 - 144 = 0$$
$$\Rightarrow \quad 9k^2 = 144 \Rightarrow k^2 = 16 \quad \therefore\ k = \pm 4$$

Example 5.94 : Find the value of k for which the equation $(k - 12)\,x^2 + 2\,(k - 12)\,x + 2 = 0$ has equal roots.

Solution : $(k - 12)x^2 + 2(k - 12)\,x + 2 = 0$

Here $a = k - 12$, $b = 2\,(k - 12)$, $c = 2$.

Given : The equation has equal roots

$$\therefore\ \Delta = 0$$
$$\text{(i.e.,)}\ b^2 - 4ac = 0$$
$$\Rightarrow\ [2\,(k - 12)]^2 - 4\,(k - 12)\,(2) = 0$$
$$\Rightarrow\ 4(k - 12)\,(k - 14) = 0$$
$$k - 12 = 0 \quad \text{or } k - 14 = 0$$
$$\Rightarrow \quad k = 12 \text{ or } k = 14$$

Example 5.95 : Find the value of k for which the equation $x^2 - 4x + k = 0$ has distinct real roots.

Solution : $x^2 - 4x + k = 0$

Here $a = 1$, $b = -4$, $c = k$.

$$\therefore\ \Delta = b^2 - 4ac = (-4)^2 - 4.1.k = 16 - 4k$$

Given : The equation has distinct real roots

$$\Rightarrow \Delta > 0 \quad \Rightarrow 16 - 4\,k > 0$$
$$\Rightarrow -4\,k > -16 \quad \therefore\ k < 4$$

Hence the equation will have distinct roots, if $k < 4$.

EXERCISE 5.19

I. Determine the nature of roots of the following quadratic equations.

(i) $(x - 1)\,(2x - 5) = 0$ (ii) $2x^2 - 3x + 4 = 0$

(iii) $3x^2 - 2\sqrt{6}\,x + 2 = 0$ (iv) $\frac{3}{5}x^2 - \frac{2}{3}x + 1 = 0$

(v) $(x - 2a)(x - 2b) = 4ab$ (vi) $9a^2b^2x^2 - 24abcdx + 16c^2d^2 = 0,\ a \neq 0\ b \neq 0.$

II. Find the values of k for which the roots are real and equal in each of the following equation :

(i) $2x^2 - 10x + k = 0$ (ii) $12x^2 + 4kx + 3 = 0$

(iii) $2x^2 - kx + 1 = 0$ (iv) $kx^2 - 5x + k = 0$

(v) $x^2 - 2x(1 + 3k) + 7(3 + 2k) = 0$ (vi) $(k + 1)x^2 - 2(k - 1)x + 1 = 0$

III Find the values of k which the equation $kx^2 - 6x - 2 = 0$ has real roots.

IV. 1. If the equation $(1 + m^2)x^2 + 2mcx + (c^2 - a^2) = 0$ has equal roots, prove that $c^2 = a^2(1 + m^2)$.

2. If the roots of the equation $(b - c)x^2 + (c - a)x + (a - b) = 0$ are equal, then prove that $2b = a + c$.

3. If the roots of the equation $(a^2 + b^2)x^2 - 2(ac + bd)x + c^2 + d^2 = 0$ are equal, prove that $\frac{a}{b} = \frac{c}{d}$.

4. If the roots of $(a - b)x^2 + (b - c)x + (c - a) = 0$ are equal, prove that $2a = b + c$.

5.8.6 Relation between roots and coefficients

Consider the general quadratic equation $ax^2 + bx + c = 0$, Where $a, b, c \in$ R,and $a \neq 0$.

Coefficient of x^2 is a ; coefficient of x is b ; contant term is c.

We have, $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Let the two roots be α and β

Let $$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

then, Sum of the roots is $$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = -\frac{2b}{2a}$$

$$\alpha + \beta = \frac{-b}{a} = \frac{-\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

Product of the roots :

$$\alpha\beta = \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$= \frac{(-b)^2-\left(\sqrt{b^2-4ac}\right)^2}{4a^2} = \frac{b^2-(b^2-4ac)}{4a^2}$$

$$= \frac{4ac}{4a^2}$$

$$\alpha\beta = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

Formation of quadratic equation with given roots

Since α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in$ R and $a \neq 0$

$$(x - \alpha) = 0 \text{ and } (x - \beta) = 0$$

$$\therefore \quad (x - \alpha)(x - \beta) = 0$$

$$x^2 - (\alpha + \beta)\, x + \alpha\beta = 0$$

$\therefore$ The required equation is x^2 – (sum of the roots) x + (product of the roots) = 0

Example 5.96 : Find the sum and product of the equation $2x^2 - 4x + 1 = 0$

Solution : $2x^2 - 4x + 1 = 0$

Here $a = 2$, $b = -4$, $c = 1$

Sum of the roots $= \frac{-b}{a} = \frac{-(-4)}{2} = 2$

Product of the roots $= \frac{c}{a} = \frac{1}{2}$

Example 5.97 : If one root of the equation $3x^2 - 10x + 3 = 0$ is $\frac{1}{3}$, find the other root.

Solution : $3x^2 - 10x + 3 = 0$

Here $a = 3$, $b = -10$, $c = 3$.

Let the two roots be α and β

$$\therefore \alpha + \beta = \frac{-b}{a} = \frac{-(-10)}{3} = \frac{10}{3} \; ; \alpha\beta = \frac{c}{a} = \frac{3}{3} = 1$$

Suppose, $\alpha = \frac{1}{3}$

$$\therefore \frac{1}{3}\beta = 1 \Rightarrow \beta = 3 \therefore \text{ the other root is 3.}$$

Example 5.98 : If α and β are the roots of the equation $3x^2 - 5x + 2 = 0$, find the value of (i) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$, (ii) $\alpha - \beta$, (iii) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$

Solution : $3x^2 - 5x + 2 = 0$

Comparing with $ax^2 + bx + c = 0$,

$a = 3$, $b = -5$, $c = 2$

α and β are the roots of the equation.

$$\therefore \alpha + \beta = \frac{-b}{a} = \frac{-(-5)}{3} = \frac{5}{3} \text{ and } \alpha\beta = \frac{c}{a} = \frac{2}{3}$$

(i) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$ [from $(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$]

$$= \frac{\left(\frac{5}{3}\right)^2 - 2\left(\frac{2}{3}\right)}{\frac{2}{3}} = \frac{\frac{25}{9} - \frac{4}{3}}{\frac{2}{3}}$$

$$= \frac{25-12}{9} \times \frac{3}{2} = \frac{13}{9} \times \frac{3}{2} = \frac{13}{6}$$

(ii) $\alpha - \beta$

Consider $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$$= \left(\frac{5}{3}\right)^2 - 4\left(\frac{2}{3}\right)$$

$$= \frac{25}{9} - \frac{8}{3}$$

$$= \frac{25-24}{9} = \frac{1}{9}$$

$$\therefore \alpha - \beta = \pm\frac{1}{3}$$

(iii) $$\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3+\beta^3}{\alpha\beta}$$

$$= \frac{(\alpha+\beta)^3 - 3\,\alpha\beta\,(\alpha+\beta)}{\alpha\beta}$$ [from $(\alpha+\beta)^3 = \alpha^3 + \beta^3 + 3\alpha^2\beta + 3\alpha\beta^2$]

$$= \frac{\left(\frac{5}{3}\right)^3 - 3\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)}{\frac{2}{3}} = \frac{35}{18}$$ (on simplifying)

Example 5.99 : Form the equation whose roots are $7+\sqrt{3}$ and $7-\sqrt{3}$

Solution : Given roots are $7+\sqrt{3}$ and $7-\sqrt{3}$

$\therefore$ Sum of the roots $= 7+\sqrt{3} + 7-\sqrt{3} = 14$

Product of the roots $= (7+\sqrt{3})(7-\sqrt{3})$

$= (7)^2 - (\sqrt{3})^2 = 49 - 3 = 46$

The required equation is

x^2 – (sum of the roots) x + (product of the roots) = 0

$\therefore x^2 - 14x + 46 = 0$

Example 5.100 : If α and β are the roots of the equation $3x^2 - 6x + 1 = 0$, form the equation whose roots are (i) $\frac{1}{\alpha}, \frac{1}{\beta}$, (ii) $\alpha^2\beta, \beta^2\alpha$, (iii) $2\alpha+\beta, 2\beta+\alpha$

Solution : $3x^2 - 6x + 1 = 0$

$$\alpha+\beta = \frac{-b}{a} = \frac{-(-6)}{3} = 2 \text{ and } \alpha\beta = \frac{c}{a} = \frac{1}{3}$$

(i) Given roots are $\frac{1}{\alpha}, \frac{1}{\beta}$

$$\text{Sum of the roots} = \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta+\alpha}{\alpha\beta} = \frac{2}{\left(\frac{1}{3}\right)} = 2 \times 3 = 6$$

$$\text{Product of the roots} = \frac{1}{\alpha} \times \frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{1}{\left(\frac{1}{3}\right)} = 3$$

x^2 – (sum of the roots) x + (product of the roots) = 0

$x^2 - 6x + 3 = 0$ is the required equation.

(ii) Given roots are $\alpha^2\beta, \beta^2\alpha$

$$\text{Sum of the roots} = \alpha^2\beta + \beta^2\alpha$$

$$= \alpha\beta(\alpha + \beta) = \frac{1}{3}(2) = \frac{2}{3}$$

$$\text{Product of the roots} = (\alpha^2\beta)(\beta^2\alpha) = \alpha^3\beta^3 = (\alpha\beta)^3$$

$$= \left(\frac{1}{3}\right)^3 = \frac{1}{27}$$

x^2 – (sum of the roots) x + (product of the roots) = 0

(i.e.,) $x^2 - \frac{2}{3}x + \frac{1}{27} = 0$

$\Rightarrow 27x^2 - 18x + 1 = 0$ is the required equation.

(iii) Given roots are $2\alpha + \beta, 2\beta + \alpha$

$$\text{Sum of the roots} = 2\alpha + \beta + 2\beta + \alpha$$

$$= 3\alpha + 3\beta = 3(\alpha + \beta)$$

$$= 3(2) = 6$$

$$\text{Product of the roots} = (2\alpha + \beta)(2\beta + \alpha)$$

$$= 4\alpha\beta + 2\alpha^2 + 2\beta^2 + \alpha\beta$$

$$= 5\,\alpha\beta + 2\,[(\alpha+\beta)^2 - 2\alpha\beta]$$

$$= \frac{5}{3} + 2\left(4 - \frac{2}{3}\right) = \frac{25}{3}$$

x^2 – (sum of the roots) x + (product of the roots) = 0

$$\Rightarrow x^2 - 6x + \frac{25}{3} = 0$$

$\Rightarrow 3x^2 - 18x + 25 = 0$ is the required equation.

Example 5.101 : If one root of the equation $x^2 - 3x + q = 0$ is twice the other, find the value of q.

Solution : $x^2 - 3x + q = 0$

Let α and 2α be the roots.

$$\therefore\ \alpha + 2\alpha = 3$$

$$\Rightarrow\ 3\alpha = 3$$

$$\Rightarrow\ \alpha = 1$$

Also $\alpha\,.\,(2\alpha) = q$

$$2\alpha^2 = q$$

$$2 = q \qquad \text{(Since } \alpha = 1\text{)}$$

EXERCISE 5.20

1. Find the sum and product of the roots of the following equations :

   (a) $x^2 - 6x + 5 = 0$ (b) $kx^2 + rx + pk = 0$

   (c) $3x^2 - 5x = 0$ (d) $8x^2 - 25 = 0$

2. Form the equation whose roots are

   (a) 3 , 4 (b) $3 + \sqrt{7}$, $3 - \sqrt{7}$, (c) $\frac{4+\sqrt{2}}{2}$, $\frac{4-\sqrt{2}}{2}$

3. If α and β are the roots of the equation $3x^2 - 6x + 4 = 0$, find the value of $\alpha^2 + \beta^2$.

4. If α and β are the roots of the equation $2x^2 - 5x - 12 = 0$ find the value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$.

5. α and β are the roots of a quadratic equation such that $\alpha + \beta = 8$ and $\alpha - \beta = 8$. Find the quadratic equation.

6. Form the equation one of whose roots is – 5 and sum of the roots is – 2.

7. If α and β are the roots of the equation $x^2 - 3x - 4 = 0$ form the equation whose roots are (a) $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$, (b) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$.

8. If α and β are the roots of the equation $x^2 + 8x = -15$ form the equation whose roots are $(\alpha + \beta)$ and $3\alpha\beta$.

9. If α and β are the roots of the equation $x^2 - 2x + 7 = 0$, form the equation whose roots are $\alpha^2\beta$, $\alpha\beta^2$.

10. If one root of the equation $2x^2 - ax + 64 = 0$ is twice the other, find a.

11. If one root of the equation $3x^2 + kx - 81 = 0$ is the square of the other, find k.

12. If α and β are the roots of $5x^2 - px + 1 = 0$ and $\alpha - \beta = 1$ find p.

6. MATRICES

6.1. INTRODUCTION

One of the biggest problems in applying mathematics to the real world is developing a means of systematising and handling the great number of variables and large amounts of data that are inherent in a real life situation. One step in handling large amount of data is the creation of a mathematical model called **Matrices.**

Matrix algebra was originated by the English Mathematician Arthur Cayley (1821 - 1895) in his work 'A Memoir on the Theory of Matrices' published in 1858. The term 'matrix' was first used in 1850 by the English Mathematician James Joseph Sylvester (1814 - 1897) a life-long friend and collaborator of Cayley in Mathematics.

It has applications in Operations Research and Psychology. Apart from these, matrices are now indispensable in the branches of Engineering, Physical and Social Sciences, Business Management, Statistics, Modern control systems and Cryptography.

It is a powerful tool in modern mathematics. It has wide applications. Demographers use matrices in the study of births and survivals, marriage and descent, class structure and mobility, etc. Matrices are more useful for practical business purposes. The matrix form suits very well for games theory, allocation of expenses, budgeting for by-products, etc. Economists now use matrices very extensively in 'social accounting' 'input-output tables' and in the study of 'inter-industry economics'.

Let us consider the following illustration :

In an elocution contest, a participant can speak one of the five languages. Hindi, English, Punjabi, Gujarati and Tamil. A college (No. 1) sent 30 students of which 10 offered to speak in Hindi, 9 in English, 6 in Punjabi, 3 in Gujarati and the rest in Tamil. Another college (No.2) sent 25 students of which 7 speak in Hindi, 8 in English and 10 in Punjabi. Out of 22 students from the third college (No. 3) 12 offered to speak in Hindi, 5 in English and 5 in Gujarati.

The information furnished in the above manner is somewhat cumbersome. It can be written in a more compact manner if we consider the following tabular form :

	Hindi	English	Punjabi	Gujarati	Tamil
College 1	10	9	6	3	2
College 2	7	8	10	0	0
College 3	12	5	0	5	0

The number in the above data are said to form a rectangular array. In any such array, lines across the page are called **rows** and lines down the page are called **columns.** Any one number within the arrangement is called an entry or an element. Thus in the above data there are 3 rows and 5 columns and hence $3 \times 5 = 15$ elements. If it is enclosed by a pair of square brackets then

$$\begin{bmatrix} 10 & 9 & 6 & 3 & 2 \\ 7 & 8 & 10 & 0 & 0 \\ 12 & 5 & 0 & 5 & 0 \end{bmatrix} \text{ is called a matrix.}$$

Since it has 3 rows and 5 columns it is said to be matrix of order 3×5 (read as '3' by '5') matrix. It may be noted that a matrix can have any number of rows and any number of columns.

Definition : A matrix is a rectangular array of numbers arranged in rows and columns enclosed by a pair of brackets and subject to certain rules of presentation.

Sometimes a pair of brackets [ ], or a pair of double vertical lines || || are used instead of a pair of parantheses.

Example :

First column, Second column, Third column

First row, Second row

$$\begin{pmatrix} 3 & 1 & 0 \\ 2 & 5 & 6 \end{pmatrix} \text{ is a matrix.}$$

This may also be put as

$$\begin{bmatrix} 3 & 1 & 0 \\ 2 & 5 & 6 \end{bmatrix} \quad \text{(or)} \quad \begin{Vmatrix} 3 & 1 & 0 \\ 2 & 5 & 6 \end{Vmatrix}$$

Notations : A matrix is usually denoted by a capital letter and its elements by corresponding small letters followed by two subscripts, the first one indicating the row and the second one the column in which the element appears.

6.2 ORDER AND FORMATION OF MATRICES :

A matrix having m rows and n columns is called a matrix of order $m \times n$ or simply $m \times n$ matrix (read as an m by n matrix). In general an $m \times n$ matrix has the form.

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1j} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2j} & \ldots\ldots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \ldots\ldots & a_{ij} & \ldots\ldots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mj} & \ldots\ldots & a_{mn} \end{bmatrix}$$

Where $a_{11}, a_{12}, \ldots$ stand for entries of the matrix. The above matrix can also be written in a more concise form as :

$$A = [\, a_{ij} \,]_{m \times n}$$

where $i = 1, 2, 3, \ldots, m$, $j = 1, 2, 3, \ldots, n$ and where a_{ij} is the element in the i^{th} row and j^{th} column and is referred as $(i, j)^{th}$ element.

Illustration *:* Consider $A = \begin{pmatrix} 1 & 3 & 5 \\ 0 & -5 & 9 \\ 7 & 11 & -13 \end{pmatrix}$

Clearly A is 3 × 3 matrix.

Here $a_{11} = 1$ (i.e.,) the element which occurs in the first row and first column = 1.

$a_{12} = 3$ (i.e.,) the element which occurs in the first row and second column = 3.

Similarly, we have

$a_{13} = 5,\ a_{21} = 0,\ a_{22} = -5,\ a_{23} = 9,\ a_{31} = 7,\ a_{32} = 11,\ a_{33} = -13$

Example 6.1 : Construct a 3 × 2 matrix whose elements are given by $a_{ij} = i + 2j$.

Solution : The general 3 × 2 matrix is given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times 2}$$

Now, since $a_{ij} = i + 2j$, we have

$$a_{11} + 1 + 2(1) = 3 \qquad a_{12} = 1 + 2(2) = 5$$

$$a_{21} = 2 + 2(1) = 4 \qquad a_{22} = 2 + 2(2) = 6$$

$$a_{31} = 3 + 2(1) = 5 \qquad a_{32} = 3 + 2(2) = 7$$

Hence, the required matrix is given by

$$A = \begin{bmatrix} 3 & 5 \\ 4 & 6 \\ 5 & 7 \end{bmatrix}$$

Example 6.2 : A matrix has 10 elements. What are the possible orders it can have ?

Solution : We know that if a matrix is of order $m \times n$, it has mn elements. Thus to find all possible orders of a matrix with 10 elements, we will find all ordered pairs the product of whose numbers is 10.

Thus, all possible ordered pairs are (1, 10), (10, 1), (2, 5), (5, 2)

Hence, possible orders are 1×10, 10×1, 2×5, 5×2.

Example 6.3 : If $A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 6 \end{bmatrix}$ Find the order of the matrix. Also, find the elements a_{21} and a_{12}.

Solution : Since the matrix A has 2 rows and 3 columns, it is of order 2×3. The element a_{21} is the element in second row and first column which is 2. The element a_{12} is the element in first row and second column is –1.

$$\therefore a_{21} = 2, \qquad a_{12} = -1.$$

Example 6.4 : Construct a 2×2 matrix $A = [a_{ij}]$ whose elements are given by

(a) $\frac{(i-j)^2}{2}$ (b) $\frac{(2i+j)^2}{2}$ (c) $\frac{(i-2j)^2}{2}$

Solution : The general 2 × 2 matrix is given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

(a) $$a_{ij} = \frac{(i-j)^2}{2} \text{ where } 1 \le i \le 2,\ 1 \le j \le 2$$

$$a_{11} = \frac{(1-1)^2}{2} = 0$$

$$a_{12} = \frac{(1-2)^2}{2} = \frac{(-1)^2}{2} = \frac{1}{2}$$

$$a_{21} = \frac{(2-1)^2}{2} = \frac{1}{2}$$

$$a_{22} = \frac{(2-2)^2}{2} = 0$$

Hence the required matrix is $A = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$

(b) $$a_{ij} = \frac{(2i+j)^2}{2}, \text{ where } 1 \le i \le 2,\ 1 \le j \le 2$$

$$a_{11} = \frac{(2+1)^2}{2} = \frac{9}{2}; \qquad a_{12} = \frac{(2+2)^2}{2} = \frac{16}{2} = 8$$

$$a_{21} = \frac{(4+1)^2}{2} = \frac{25}{2}; \qquad a_{22} = \frac{(4+2)^2}{2} = \frac{36}{2} = 18$$

∴ The required matrix is $A = \begin{bmatrix} \frac{9}{2} & 8 \\ \frac{25}{2} & 18 \end{bmatrix}$

(c) $a_{ij} = \dfrac{(i-2j)^2}{2}$ where $1 \le i \le 2$, $1 \le j \le 2$

$a_{11} = \dfrac{(1-2)^2}{2} = \dfrac{1}{2}$; $a_{12} = \dfrac{(1-4)^2}{2} = \dfrac{9}{2}$

$a_{21} = \dfrac{(2-2)^2}{2} = 0$; $a_{22} = \dfrac{(2-4)^2}{2} = 2$

Hence, the required matrix A = $\begin{bmatrix} \frac{1}{2} & \frac{9}{2} \\ 0 & 2 \end{bmatrix}$

EXERCISE - 6.1

1. Find the order of the following matrices :

(a) $\begin{pmatrix} 2 & 3 & 5 \\ 3 & 6 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -1 & 2 & 4 \\ 3 & 6 & 0 & 5 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & -2 \\ 3 & 0 \\ 7 & -8 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 3 & -3 & 1 \end{pmatrix}$

2. If a matrix has 8 elements, what are the possible orders it can have ? What if it has 5 elements ?

3. Construct a 3 × 3 matrix whose elements are given by

(a) $a_{ij} = i \times j$ (b) $a_{ij} = \dfrac{i}{j}$ (c) $a_{ij} = 2i - j$

4. Construct a 3 × 2 matrix whose elements are given by

(a) $a_{ij} = 3i \times 2j$ (b) $a_{ij} = i - 2j$ (c) $a_{ij} = ij$

(d) $a_{ij} = \dfrac{3i-j}{2}$ (e) $a_{ij} = \dfrac{i+4j}{2}$ (f) $a_{ij} = i^2j^2$

5. Let A = $[a_{ij}] = \begin{bmatrix} 1 & 4 & -1 \\ 2 & 6 & 5 \end{bmatrix}$ Find the order of the matrix. Also find the elements a_{23} and a_{12}.

6. Construct a 4 × 3 matrix whose elements are (a) $a_{ij} = 2i + \dfrac{i}{j}$; (b) $a_{ij} = \dfrac{i-j}{i+j}$;

(c) $a_{ij} = i$; (d) $a_{ij} = \dfrac{|2i-3j|}{2}$; (e) $a_{ij} = \dfrac{|-3i+j|}{2}$

6.3 TYPES OF MATRICES

In this sub-section, we shall define certain types of matrices.

1. Row matrix : A matrix is said to be a row matrix if it has only one row and any number of columns.

For example $[5 \quad 0 \quad 3]$ is a row matrix of order 1×3.

2. Column matrix : A matrix is said to be a column matrix if it has only one column and any number of rows.

For example $\begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}$ is a column matrix of order 3×1.

3. Square matrix : A matrix in which the number of rows is equal to the number of columns, is called a square matrix. Thus a $m \times n$ matrix will be a square matrix if $m = n$.

Thus $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2\times2}$, $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}_{3\times3}$ are square matrices.

For example, $\begin{bmatrix} 2 & 3 \\ 5 & 0 \end{bmatrix}$, $[5]$ are all square matrices of orders 2 and 1 respectively.

4. Zero matrix or null matrix : A matrix is said to be a zero matrix or null matrix if all its elements are zero.

For example $[0]$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are all zero matrices, but of different orders. We denote zero matrix by 0.

5. Diagonal Matrix : A square matrix, all of whose elements, except those in the leading diagonal are zero is called a diagonal matrix.

(i.e.,) A square matrix $A = [a_{ij}]$ is said to be diagonal matrix if $a_{ij} = 0$ for $i \neq j$.

For example,

$$\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$ are diagonal matrices of order 2 and 3 respectively.

6. Scalar matrix : A scalar matrix is a diagonal matrix whose principal diagonal (or) leading diagonal (or) main diagonal elements are equal.

Example : $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is a scalar matrix of order 2.

$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a scalar matrix of order 3.

7. Identity matrix : A square matrix in which elements in the main diagonal are all '1' and rest are all zero is called an identity matrix or unit matrix.

Thus, the square matrix A = $[a_{ij}]$, is an identity matrix if

$$a_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

A unit matrix of order n is written as I_n.

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, are identity matrices of order 2 and 3 respectively.

8. Transpose of a matrix : The matrix obtained by interchanging rows and columns of the matrix A is called the transpose of A and is denoted by A' or A^T (read as A transpose)

For example,

If $\quad A = \begin{pmatrix} 3 & 2 \\ 4 & 1 \\ 7 & -5 \end{pmatrix}_{3\times2}$ then $A^T = \begin{pmatrix} 3 & 4 & 7 \\ 2 & 1 & -5 \end{pmatrix}_{2\times3}$

Symbolically if

$$A = (a_{ij})_{m \times n} \text{ then } A^T = (a_{ji})_{n \times m}$$

9. Triangular matrix : A square matrix in which all the entries above the main diagonal are zero is called a **lower triangular matrix.** If all the entries below the main diagonal are zero, it is called an **upper triangular matrix**.

A square matrix $A = (a_{ij})_{n \times n}$ is called **upper triangular matrix** if $a_{ij} = 0$ for $i > j$ and is called **lower triangular matrix** if $a_{ij} = 0$ for $i < j$.

For example, $A = \begin{bmatrix} 1 & 3 & -7 \\ 0 & 2 & 6 \\ 0 & 0 & -1 \end{bmatrix}$ is an upper triangular matrix and

$B = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ -7 & 6 & -1 \end{bmatrix}$ is an lower triangular matrix.

6.4 OPERATIONS ON MATRICES

In this section, we shall discuss about equality of matrices, multiplication of a matrix by a scalar, addition, subtraction and multiplication of matrices.

Equality of matrices :

Two matrices are said to be equal iff

(i) They are of the same order and

(ii) each element of one is equal to the corresponding element of the other.
(i.e.,) if $A = [a_{ij}]_{m \times n}$ $B = [b_{ij}]_{m \times n}$ then A = B iff $a_{ij} = b_{ij}$ $\forall$ i = 1, 2, ..., m

$j = 1, 2, ..., n$

Example : If $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$

Then $a = 2$, $b = -1$, $c = 4$, $d = 3$.

Multiplication of a matrix by a scalar (Scalar Multiplication)

Let $A = [a_{ij}]$ be an $m \times n$ matrix and k be any number called a scalar. Then the matrix obtained by multiplying every element of A by k is called the scalar multiple of A by k and is denoted by kA. Thus $k\,A = [ka_{ij}]_{m \times n}$.

Example : If $A = \begin{bmatrix} 2 & 1 & -3 \\ 3 & 0 & 2 \end{bmatrix}$ then $2A = 2\begin{bmatrix} 2 & 1 & -3 \\ 3 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & -6 \\ 6 & 0 & 4 \end{bmatrix}$

Negative of a matrix :

Let A be any matrix. The negative of a matrix A is – A and is obtained by changing the sign of all the entries of matrix A.

Example : If $A = \begin{bmatrix} 2 & 1 & 3 \\ -3 & 4 & -6 \end{bmatrix}$ then $-A = \begin{bmatrix} -2 & -1 & -3 \\ 3 & -4 & 6 \end{bmatrix}$

Also $-3A = -3\begin{bmatrix} 2 & 1 & 3 \\ -3 & 4 & -6 \end{bmatrix} = \begin{bmatrix} -6 & -3 & -9 \\ 9 & -12 & 18 \end{bmatrix}$

Addition of Matrices

Let A, B be two matrices, each of order $m \times n$. Then their sum A + B is a matrix of order $m \times n$ and is obtained by adding the corresponding elements of A and B.

Thus, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are two matrices of the same order, their sum A + B is defined to be the matrix of order $m \times n$ such that

$A + B = [a_{ij} + b_{ij}]$ for $i = 1, 2, ..., m$ and $j = 1, 2, 3, ..., n$.

Note : The sum of two matrices is defined only when they are of the same order.

Example : If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix}$

$A + B = \begin{bmatrix} 1+2 & 2+1 & 3+4 \\ 0+3 & 2+2 & 1+1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 7 \\ 3 & 4 & 2 \end{bmatrix}$

Example : If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & -1 \\ 0 & 5 & -1 \end{bmatrix}$, then A + B is not defined, because A and B are not of the same order.

Subtraction of Matrices : For two matrices A and B of the same order, we define A – B = A + (–B).

Example : If A = $\begin{bmatrix} -3 & -2 & 1 \\ 1 & -4 & 7 \end{bmatrix}$ and B = $\begin{bmatrix} 0 & 1 & 5 \\ 2 & 0 & -4 \end{bmatrix}$

$$A - B = A + (-B) = \begin{bmatrix} -3 & -2 & 1 \\ 1 & -4 & 7 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -5 \\ -2 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & -3 & -4 \\ -1 & -4 & 11 \end{bmatrix}$$

***Example 6.5* :** If A = $\begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} 7 & 6 & 3 \\ 1 & 4 & 5 \end{bmatrix}$ find the value of 2A + 3B, A – B.

Solution :

$$2A = 2\begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 6 \\ 4 & 2 & 8 \end{bmatrix}$$

$$3B = 3\begin{bmatrix} 7 & 6 & 3 \\ 1 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 21 & 18 & 9 \\ 3 & 12 & 15 \end{bmatrix}$$

$$2A + 3B = \begin{bmatrix} 0 & 4 & 6 \\ 4 & 2 & 8 \end{bmatrix} + \begin{bmatrix} 21 & 18 & 9 \\ 3 & 12 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 22 & 15 \\ 7 & 14 & 23 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} - \begin{bmatrix} 7 & 6 & 3 \\ 1 & 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} + \begin{bmatrix} -7 & -6 & -3 \\ -1 & -4 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & -4 & 0 \\ 1 & -3 & -1 \end{bmatrix}$$

Multiplication of Matrices

Two matrices A and B are conformable for the product AB if the number of columns in A (pre-multiplier) is same as the number of rows in B (post-multiplier). Thus, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ are two matrices of order $m \times n$ and $n \times p$ respectively then their product AB is of order $m \times p$ and is obtained by multiplying every row of matrix A with the corresponding elements of every column of matrix B element-wise and add the results. This method is known as **row-by-column multiplication rule.**

$$\text{Let } A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 1 & 3 \end{bmatrix}_{2\times 3} ; \; B = \begin{bmatrix} 2 & 0 & 5 \\ 5 & 1 & 2 \\ -3 & 2 & 1 \end{bmatrix}_{3\times 3}$$

Matrices A and B are conformable,

i.e., the product matrix AB can be found.

$$AB = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 5 \\ 5 & 1 & 2 \\ -3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1 \;\; 4 \;\; 5)\begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} & (1 \;\; 4 \;\; 5)\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} & (1 \;\; 4 \;\; 5)\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} \\ (2 \;\; 1 \;\; 3)\begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} & (2 \;\; 1 \;\; 3)\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} & (2 \;\; 1 \;\; 3)\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} (1)(2)+(4)(5)+(5)(-3) & (1)(0)+(4)(1)+(5)(2) & (1)(5)+(4)(2)+(5)(1) \\ (2)(2)+(1)(5)+(3)(-3) & (2)(0)+(1)(1)+(3)(2) & (2)(5)+(1)(2)+(3)(1) \end{bmatrix}$$

$$\text{Thus } AB = \begin{bmatrix} 7 & 14 & 18 \\ 0 & 7 & 15 \end{bmatrix}_{2\times 3}$$

Example 6.6 : Find AB and BA if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix}$

Solution : We observe that A is a 2 × 2 matrix and B is a 2 × 2 matrix. Hence, AB and BA are both defined and are of orders 2 × 2 respectively.

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} (1\ \ 2)\begin{pmatrix}-1\\4\end{pmatrix} & (1\ \ 2)\begin{pmatrix}3\\2\end{pmatrix} \\ (3\ \ 4)\begin{pmatrix}-1\\4\end{pmatrix} & (3\ \ 4)\begin{pmatrix}3\\2\end{pmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} (1\times(-1))+(2\times 4) & (1\times 3)+(2\times 2) \\ (3\times(-1))+(4\times 4) & (3\times 3)+(4\times 2) \end{bmatrix}$$

$$= \begin{bmatrix} -1+8 & 3+4 \\ -3+16 & 9+8 \end{bmatrix} = \begin{bmatrix} 7 & 7 \\ 13 & 17 \end{bmatrix}$$

$$BA = \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} (-1\times 1)+(3\times 3) & (-1\times 2)+(3\times 4) \\ (4\times 1)+(2\times 3) & (4\times 2)+(2\times 4) \end{bmatrix}$$

$$= \begin{bmatrix} -1+9 & -2+12 \\ 4+6 & 8+8 \end{bmatrix} = \begin{bmatrix} 8 & 10 \\ 10 & 16 \end{bmatrix}$$

Example 6.7 : If $A = \begin{bmatrix} 1 & 0 \\ 3 & -1 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 6 & 0 \end{bmatrix}$ will AB be equal to BA? Also find AB and BA.

Solution : We observe that A is a 3 × 2 matrix and B is a 2 × 3 matrix. Hence AB and BA are both defined and are of orders 3 × 3 and 2 × 2 respectively. Since AB and BA are of different orders. AB ≠ BA.

$$AB = \begin{bmatrix} 1 & 0 \\ 3 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 1 & -2 \\ 4 & 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (1\times 3)+(0\times 4) & (1\times 1)+(0\times 6) & (1\times(-2))+(0\times 0) \\ (3\times 3)+(-1\times 4) & (3\times 1)+(-1\times 6) & (3\times(-2))+(-1\times 0) \\ (2\times 3)+(4\times 4) & (2\times 1)+(4\times 6) & (2\times(-2))+(4\times 0) \end{bmatrix}$$

$$= \begin{bmatrix} 3+0 & 1+0 & -2+0 \\ 9-4 & 3-6 & -6+0 \\ 6+16 & 2+24 & -4+0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -2 \\ 5 & -3 & -6 \\ 22 & 26 & -4 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 6 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & -1 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} (3\times1)+(1\times3)+(-2\times2) & (3\times0)+(1\times(-1))+(-2\times4) \\ (4\times1)+(6\times3)+(0\times2) & (4\times0)+(6\times(-1))+(0\times4) \end{bmatrix}$$

$$= \begin{bmatrix} 3+3-4 & 0-1-8 \\ 4+18+0 & 0-6+0 \end{bmatrix} = \begin{bmatrix} 2 & -9 \\ 22 & -6 \end{bmatrix}$$

6.5 ALGEBRAIC FACTS

I. Properties of Matrix Addition :

(a) Matrix addition is commutative :

If A and B are any two matrices of order $m \times n$ each, then A + B = B + A.

(b) Matrix addition is associative :

If A, B and C are any three matrices of the same order, then (A + B) + C = A + (B + C).

(c) Existence of Additive Identity :

The null matrix is the identity for matrix addition. Let A be any matrix.

A + 0 = 0 + A = A where 0 is the null matrix.

(d) Existence of Additive Inverse :

If A be any given matrix then the matrix – A which must exist, is the **additive inverse** of A.

A + (–A) = (–A) + A = 0

II. Properties due to Matrix Multiplication

(a) Matrix multiplication is not commutative in general

If A is of the type $m \times n$ and B of the type $n \times p$ then AB is defined but BA is not defined. Even if AB and BA are both defined, it is not necessary that they are equal. Hence in general, AB $\neq$ BA.

(b) Matrix multiplication is distributive over matrix addition

(i) If A, B, C are $m \times n$, $n \times p$ and $n \times p$ matrices respectively, then

A (B + C) = AB + AC

(ii) If A, B, C are $m \times n$, $m \times n$ and $n \times p$ matrices respectively, then

(A + B) C = AC + BC

(c) Matrix multiplication is always associative

If A, B, C are $m \times n$, $n \times p$ and $p \times q$ matrices respectively, then

(AB)C = A (BC).

(d) Multiplication of a matrix by a unit matrix.

If A is a square matrix of order $n \times n$ and I is the unit matrix of same order then AI = IA = A.

Note : AB = 0 (null) does not necessarily imply that A = 0 or B = 0 or both = 0.

Example : A = $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \neq 0$ and B = $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \neq 0$

But AB = $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1-1 & 1-1 \\ -1+1 & -1+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

SOME MORE EXAMPLES

Example 6.8 : Find p, q, r, s if $\begin{pmatrix} -2 & p & 4 & 0 \\ 3 & 2 & q & 3 \end{pmatrix} = \begin{pmatrix} -2 & 3 & r & 0 \\ 3 & 2 & 1 & s \end{pmatrix}$

Solution : When two matrices of the same order are equal, their corresponding entries are equal.

$\Rightarrow p = 3, q = 1, r = 4, s = 3.$

Example 6.9 : Find the unknowns *a, b, c, d, x, y* in the following matrix equations.

(a) $\begin{pmatrix} d+1 & 10+a \\ 3b-2 & a-4 \end{pmatrix} = \begin{pmatrix} 2 & 2a+1 \\ b-5 & 4c \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 4 \\ x^2 & y^2 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 4 & 9 \end{pmatrix}$

Solution : (a) Equating the corresponding elements of the two matrices, we have

$d + 1 = 2 \qquad\qquad 10 + a = 2a + 1$

$\Rightarrow d = 2 - 1 \qquad\qquad a - 2a = 1 - 10$

$\Rightarrow d = 1 \qquad\qquad -a = -9$

$a = 9$

$\Rightarrow 3b - 2 = b - 5$

$3b - b = -3$

$2b = -3$

$b = -\frac{3}{2}$

$\Rightarrow a - 4 = 4c \qquad\qquad ... (1)$

putting $a = 9$ in (1)

$9 - 4 = 4c$

$c = \frac{5}{4}$

Hence, $a = 9,\ b = -\frac{3}{2},\ c = \frac{5}{4},\ d = 1.$

(b) Equating the corresponding elements of the two matrices, we have

$x^2 = 4 \qquad\qquad y^2 = 9$

$\Rightarrow x = \pm 2 \qquad\qquad \Rightarrow y = \pm 3$

Hence, $x = \pm 2,\ \ y = \pm 3$

Example 6.10 : If $A = \begin{pmatrix} 2 & 0 & 4 \\ 6 & 2 & 8 \\ 2 & 4 & 6 \end{pmatrix}, B = \begin{pmatrix} 8 & 4 & -2 \\ 0 & -2 & 0 \\ 2 & 2 & 6 \end{pmatrix}, C = \begin{pmatrix} 8 & 2 & 0 \\ 0 & 2 & -6 \\ -8 & 4 & -10 \end{pmatrix}$

Compute the following :

(a) A + B (b) A – B (c) (A – B) + C

(d) 3A + 2B – 2C (e) A – B – C

Solution : (a) $A + B = \begin{pmatrix} 2 & 0 & 4 \\ 6 & 2 & 8 \\ 2 & 4 & 6 \end{pmatrix} + \begin{pmatrix} 8 & 4 & -2 \\ 0 & -2 & 0 \\ 2 & 2 & 6 \end{pmatrix} = \begin{pmatrix} 10 & 4 & 2 \\ 6 & 0 & 8 \\ 4 & 6 & 12 \end{pmatrix}$

(b) $A - B = \begin{pmatrix} 2 & 0 & 4 \\ 6 & 2 & 8 \\ 2 & 4 & 6 \end{pmatrix} - \begin{pmatrix} 8 & 4 & -2 \\ 0 & -2 & 0 \\ 2 & 2 & 6 \end{pmatrix}$

$$= \begin{pmatrix} 2 & 0 & 4 \\ 6 & 2 & 8 \\ 2 & 4 & 6 \end{pmatrix} + \begin{pmatrix} -8 & -4 & 2 \\ 0 & 2 & 0 \\ -2 & -2 & -6 \end{pmatrix} = \begin{pmatrix} -6 & -4 & 6 \\ 6 & 4 & 8 \\ 0 & 2 & 0 \end{pmatrix}$$

(c) (A – B) + C

We know that $A - B = \begin{pmatrix} -6 & -4 & 6 \\ 6 & 4 & 8 \\ 0 & 2 & 0 \end{pmatrix}$

$$\therefore (A - B) + C = \begin{pmatrix} -6 & -4 & 6 \\ 6 & 4 & 8 \\ 0 & 2 & 0 \end{pmatrix} + \begin{pmatrix} 8 & 2 & 0 \\ 0 & 2 & -6 \\ -8 & 4 & -10 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -2 & 6 \\ 6 & 6 & 2 \\ -8 & 6 & -10 \end{pmatrix}$$

(d) 3A + 2B – 3C

Solution : $3A+2B-3C = 3\begin{pmatrix} 2 & 0 & 4 \\ 6 & 2 & 8 \\ 2 & 4 & 6 \end{pmatrix} + 2\begin{pmatrix} 8 & 4 & -2 \\ 0 & -2 & 0 \\ 2 & 2 & 6 \end{pmatrix} - 3\begin{pmatrix} 8 & 2 & 0 \\ 0 & 2 & -6 \\ -8 & 4 & -10 \end{pmatrix}$

$$= \begin{pmatrix} 6 & 0 & 12 \\ 18 & 6 & 24 \\ 6 & 12 & 18 \end{pmatrix} + \begin{pmatrix} 16 & 8 & -4 \\ 0 & -4 & 0 \\ 4 & 4 & 12 \end{pmatrix} + \begin{pmatrix} -24 & -6 & 0 \\ 0 & -6 & 18 \\ 24 & -12 & 30 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 2 & 8 \\ 18 & -4 & 42 \\ 34 & 4 & 60 \end{pmatrix}$$

(e) A – B – C

Solution :

$$A - B - C = \begin{pmatrix} 2 & 0 & 4 \\ 6 & 2 & 8 \\ 2 & 4 & 6 \end{pmatrix} - \begin{pmatrix} 8 & 4 & -2 \\ 0 & -2 & 0 \\ 2 & 2 & 6 \end{pmatrix} - \begin{pmatrix} 8 & 2 & 0 \\ 0 & 2 & -6 \\ -8 & 4 & -10 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 4 \\ 6 & 2 & 8 \\ 2 & 4 & 6 \end{pmatrix} + \begin{pmatrix} -8 & -4 & 2 \\ 0 & 2 & 0 \\ -2 & -2 & -6 \end{pmatrix} + \begin{pmatrix} -8 & -2 & 0 \\ 0 & -2 & 6 \\ 8 & -4 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} -14 & -6 & 6 \\ 6 & 2 & 14 \\ 8 & -2 & 10 \end{pmatrix}$$

Example 6.11 : If $A = \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 2 \\ 4 & -1 \end{pmatrix}$. Find the matrix C if 2C = A + B.

Solution : $A + B = \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix} + \begin{pmatrix} -3 & 2 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 9 \\ 6 & 4 \end{pmatrix}$

$$\Rightarrow 2C = \begin{pmatrix} 0 & 9 \\ 6 & 4 \end{pmatrix}$$

$$C = \frac{1}{2}\begin{pmatrix} 0 & 9 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} 0 & \frac{9}{2} \\ 3 & 2 \end{pmatrix}$$

Example 6.12 : If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ $C = \begin{pmatrix} k & l \\ m & n \end{pmatrix}$.

Verify that A (B + C) = AB + AC

Solution :

$$B + C = \begin{pmatrix} p & q \\ r & s \end{pmatrix} + \begin{pmatrix} k & l \\ m & n \end{pmatrix}$$

$$= \begin{pmatrix} p+k & q+l \\ r+m & s+n \end{pmatrix}$$

$$A(B + C) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p+k & q+l \\ r+m & s+n \end{pmatrix}$$

$$A(B + C) = \begin{pmatrix} ap+ak+br+bm & aq+al+bs+bn \\ cp+ck+dr+dm & cq+cl+ds+dn \end{pmatrix} \quad \text{... (1)}$$

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap+br & aq+bs \\ cp+rd & cq+ds \end{pmatrix}$$

$$AC = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} k & l \\ m & n \end{pmatrix}$$

$$= \begin{pmatrix} ak+bm & al+bn \\ ck+dm & cl+dn \end{pmatrix}$$

$$AB + AC = \begin{pmatrix} ap+br & aq+bs \\ cp+rd & cq+ds \end{pmatrix} + \begin{pmatrix} ak+bm & al+bn \\ ck+dm & cl+dn \end{pmatrix}$$

$$AB + AC = \begin{pmatrix} ap+ak+br+bm & aq+al+bn+bs \\ cp+ck+dr+dm & cq+cl+dn+ds \end{pmatrix} \quad ...(2)$$

From (1) and (2) we get

A (B + C) = AB + AC. Hence the result.

Example 6.13 : If $A = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ Show that $A^2 - 7A + 10I_3 = 0$

Solution : Here $A = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$$A^2 = AA = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times3+2\times1+0\times0 & 3\times2+2\times4+0\times0 & 3\times0+2\times0+0\times5 \\ 1\times3+4\times1+0\times0 & 1\times2+4\times4+0\times0 & 1\times0+4\times0+0\times5 \\ 0\times3+0\times1+5\times0 & 0\times2+0\times4+5\times0 & 0\times0+0\times0+5\times5 \end{bmatrix}$$

$$= \begin{bmatrix} 9+2+0 & 6+8+0 & 0+0+0 \\ 3+4+0 & 2+16+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+25 \end{bmatrix} = \begin{bmatrix} 11 & 14 & 0 \\ 7 & 18 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

$\therefore A^2 - 7A + 10I_3$

$$= \begin{bmatrix} 11 & 14 & 0 \\ 7 & 18 & 0 \\ 0 & 0 & 25 \end{bmatrix} - 7 \begin{bmatrix} 3 & 2 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} + 10 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Which is true.

Example 6.14 : If $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}$

does $(A + B)^2 = A^2 + 2AB + B^2$ hold ?

Solution :

$$A + B = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 \\ -2 & 3 \end{bmatrix}$$

$$(A + B)^2 = \begin{bmatrix} 3 & 3 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} (3\times3)+(3\times(-2)) & (3\times3)+(3\times3) \\ (-2\times3)+(3\times(-2)) & (-2\times3)+(3\times3) \end{bmatrix}$$

$$= \begin{bmatrix} 9-6 & 9+9 \\ -6-6 & -6+9 \end{bmatrix}$$

$$(A + B)^2 = \begin{bmatrix} 3 & 18 \\ -12 & 3 \end{bmatrix} \quad \text{... (1)}$$

$$A^2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} (2\times2)+(-1\times-1) & (2\times-1)+(-1\times2) \\ (-1\times2)+(2\times-1) & (-1\times-1)+(2\times2) \end{bmatrix}$$

$$= \begin{bmatrix} 4+1 & -2-2 \\ -2-2 & 1+4 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (2\times1)+(-1\times-1) & (2\times4)+(-1\times1) \\ (-1\times1)+(2\times-1) & (-1\times4)+(2\times1) \end{bmatrix}$$

$$= \begin{bmatrix} 2+1 & 8-1 \\ -1-2 & -4+2 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ -3 & -2 \end{bmatrix}$$

$$2AB = 2\begin{bmatrix} 3 & 7 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} 6 & 14 \\ -6 & -4 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1\times1)+(4\times(-1)) & (1\times4)+(4\times1) \\ (-1\times1)+(1\times(-1)) & (-1\times4)+(1\times1) \end{bmatrix}$$

$$= \begin{bmatrix} 1-4 & 4+4 \\ -1-1 & -4+1 \end{bmatrix} = \begin{bmatrix} -3 & 8 \\ -2 & -3 \end{bmatrix}$$

$$A^2 + 2AB + B^2 = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix} + \begin{bmatrix} 6 & 14 \\ -6 & -4 \end{bmatrix} + \begin{bmatrix} -3 & 8 \\ -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 5+6-3 & -4+14+8 \\ -4-6-2 & 5-4-3 \end{bmatrix}$$

$$A^2 + 2AB + B^2 = \begin{bmatrix} 8 & 18 \\ -12 & -2 \end{bmatrix} \qquad \text{... (2)}$$

From (1) and (2) we see that

$(A + B)^2 = A^2 + 2AB + B^2$ <u>does not hold.</u>

(i.e.,) $(A + B)^2 \neq A^2 + 2AB + B^2$

Example 6.15 : If $A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ $B = \begin{pmatrix} -1 & -3 \\ -4 & -4 \end{pmatrix}$

Verify that $(AB)^T = B^T A^T$.

Solution : $AB = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} -1 & -3 \\ -4 & -4 \end{pmatrix}$

$$= \begin{pmatrix} (2)(-1)+(3)(-4) & (2)(-3)+(3)(-4) \\ (4)(-1)+(5)(-4) & (4)(-3)+(5)(-4) \end{pmatrix}$$

$$= \begin{pmatrix} -2-12 & -6-12 \\ -4-20 & -12-20 \end{pmatrix} = \begin{pmatrix} -14 & -18 \\ -24 & -32 \end{pmatrix}$$

$$\text{L.H.S.} = (AB)^T = \begin{pmatrix} -14 & -18 \\ -24 & -32 \end{pmatrix}^T = \begin{pmatrix} -14 & -24 \\ -18 & -32 \end{pmatrix} \quad \dots (1)$$

$$A^T = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}^T = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}$$

$$B^T = \begin{pmatrix} -1 & -3 \\ -4 & -4 \end{pmatrix}^T = \begin{pmatrix} -1 & -4 \\ -3 & -4 \end{pmatrix}$$

$$\text{R.H.S.} = B^T A^T = \begin{pmatrix} -1 & -4 \\ -3 & -4 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} (-1)(2)+(-4)(3) & (-1)(4)+(-4)(5) \\ (-3)(2)+(-4)(3) & (-3)(4)+(-4)(5) \end{pmatrix} = \begin{pmatrix} -2-12 & -4-20 \\ -6-12 & -12-20 \end{pmatrix}$$

$$\therefore B^T A^T = \begin{pmatrix} -14 & -24 \\ -18 & -32 \end{pmatrix} \dots (2)$$

From (1) and (2) L.H.S. = R.H.S.

$\therefore (AB)^T = B^T A^T$.

Example 6.16 : Prove that $(x \quad y) \begin{pmatrix} a & h \\ h & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2hxy + by^2$

Solution : L.H.S. $= (x \quad y)_{1\times 2} \left\{ \begin{pmatrix} a & h \\ h & b \end{pmatrix}_{2\times 2} \begin{pmatrix} x \\ y \end{pmatrix}_{2\times 1} \right\}$

$$= (x \quad y)_{1\times 2} \begin{pmatrix} ax+hy \\ hx+by \end{pmatrix}_{2\times 1}$$

$= [x\,(ax + hy) + y\,(hx + by)]_{1\times 1}$

$= ax^2 + hxy + hxy + by^2$

$= ax^2 + 2hxy + by^2$ = R.H.S.

Example 6.17 : Solve for x, y

$$\begin{pmatrix} x^2 \\ y^2 \end{pmatrix} + 2\begin{pmatrix} 2x \\ -3y \end{pmatrix} = \begin{pmatrix} 5 \\ -8 \end{pmatrix}$$

Solution :

$$\begin{pmatrix} x^2 + 4x \\ y^2 - 6y \end{pmatrix} = \begin{pmatrix} 5 \\ -8 \end{pmatrix}$$

Now $x^2 + 4x = 5 \quad \Rightarrow \quad x^2 + 4x - 5 = 0$

$(x + 5)(x - 1) = 0$

$x = -5$ or 1

$y^2 - 6y = -8 \quad \Rightarrow \quad y^2 - 6y + 8 = 0$

$\Rightarrow \quad (y - 4)(y - 2) = 0$

$y = 4$, $y = 2$

$\therefore\ y = 4$ or 2.

$\therefore$ The values of x and y are $x = -5$ or 1, $y = 4$ or 2.

Example 6.18 : Solve : $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

Solution : $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ x + 2y \end{pmatrix}$

By equation $\begin{pmatrix} 2x + y \\ x + 2y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

$\therefore\ 2x + y = 4\ ;\ x + 2y = 5$

rewriting, $2x + y - 4 = 0 \quad \ldots (1)$

$x + 2y - 5 = 0 \quad \ldots (2)$

using cross multiplication rule, we get

$$\begin{array}{ccccccc} & x & & y & & 1 & \\ 1 & & -4 & & 2 & & 1 \\ 2 & & -5 & & 1 & & 2 \end{array}$$

$$\frac{x}{-5+8} = \frac{y}{-4+10} = \frac{1}{4-1}$$

$$x = \frac{3}{3}, y = \frac{6}{3}$$

$$\therefore x = 1, y = 2.$$

EXERCISE - 6.2

1. If A is of order $m \times n$ and B is of order $n \times p$ what is the order of the product matrix AB, BA ?
2. A has x rows and $x + 5$ columns, B has y rows and 11-y columns. Both the products AB and BA exist. Find x, y.
3. If A_{mn} and B_{pq} are two matrices (the subscripts indicate their orders), state the condition when they are conformable for

   (i) addition, (ii) multiplication, (iii) both addition and multiplication.
4. Find the order of the product matrix AB if :

	(i)	(ii)	(iii)	(iv)
Type of A	3 × 4	4 × 3	6 × 7	1 × 4
Type of B	4 × 5	3 × 4	7 × 2	4 × 1

5. When are two matrices said to be equal ?
6. If $\begin{pmatrix} x & x-y \\ x+y & x-z \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ 10 & 2 \end{pmatrix}$ find x, y, z
7. Find the values of x, y, z from the matrix equation.

   (a) $\begin{pmatrix} x-3 & 3x-z \\ x+y+2 & x+y+z \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 5 & 6 \end{pmatrix}$ (b) $[x \quad y+z \quad z-3] + [y \quad 4 \quad 5] = [4 \quad 9 \quad 12]$
8. If A = $\begin{pmatrix} 9 & 1 \\ 4 & 3 \end{pmatrix}$ and B = $\begin{pmatrix} 1 & 5 \\ 7 & 12 \end{pmatrix}$ find the matrix X such that 3A + 5B + 2X = 0.
9. If A = $\begin{pmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{pmatrix}$ and B = $\begin{pmatrix} 7 & 6 & 3 \\ 1 & 4 & 5 \end{pmatrix}$ find the value of (a) B – 4A, (b) 5A – 2B.
10. Find X and Y if 2X + Y = $\begin{bmatrix} 4 & 4 & 7 \\ 7 & 3 & 4 \end{bmatrix}$ and X – 2Y = $\begin{bmatrix} -3 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$
11. Find X and Y if X + Y = $\begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and X – Y = $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$.

12. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ show that $A^2 - (a + d)A = (bc - ad)I$

13. If $A = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} a & 1 \\ b & -1 \end{pmatrix}$ and $(A + B)^2 = A^2 + B^2$ find a and b.

14. Given the matrices A, B, C

$A = \begin{pmatrix} 2 & 3 & -1 \\ 3 & 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $C = (1 \quad -2)$. Verify that $(AB)C = A(BC)$

15. If $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & -3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 & 6 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & -2 & 1 \\ -1 & 2 & 3 \\ -1 & -2 & 2 \end{bmatrix}$.

Verify that $A(B + C) = AB + AC$.

16. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ show that $A^2 - 5A + 7I_2 = 0$

17. Solve the matrix equation $[x \quad 1] \begin{bmatrix} 1 & 0 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ 5 \end{bmatrix} = [0]$

18. Verify that $(AB)^T = B^T A^T$ if $A = \begin{pmatrix} 2 & 3 & -1 \\ 4 & 1 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 \\ 3 & -3 \\ 2 & 6 \end{pmatrix}$

19. Solve : $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

20. Solve : $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

21. Find X, given $X + \begin{pmatrix} 5 & 6 & -1 \\ -2 & 3 & -4 \end{pmatrix} = \begin{pmatrix} 6 & 4 & -2 \\ -3 & -1 & -4 \end{pmatrix}$

22. Find Y, given $\begin{pmatrix} -3 & -2 & 4 \\ 5 & -6 & -3 \end{pmatrix} - 2Y = \begin{pmatrix} -5 & 4 & 6 \\ 3 & -4 & 5 \end{pmatrix}$

7. THEORETICAL GEOMETRY

INTRODUCTION

We have learnt geometry in the earlier standards as a branch of Mathematics concerned with the properties and relationships between points, lines, planes and figures. This geometry is nothing but Theoretical, in which we speak about ***geometrical shapes on their general characters without actual measurements.*** There is another branch of geometry, which deals with geometrical shapes with their exact measurements and is referred as ''Practical Geometry''. This is discussed elaborately under Chapter 10 in this book. ''Co-ordinate Geometry'' is another branch of geometry in which geometric relations are expressed in algebraic form as discussed in Chapter 8 in this book.

Theoretical Geometry :

Definition : The geometry which deals with properties and characters of various geometrical shapes with axioms/theorems without accurate measurements is known as Theoretical Geometry.

Explanation : The relationship between points, lines, plane figures and solid figures are obtained through theoretical geometry and it is explained using axioms (without proof) and theorems (with proof). In this process, the length, area and volume of the objects are derived with a rough sketch without exact measurements.

Applications : This being the basic concept in geometry, is applied practically in other branches of geometry like Analytical geometry, Practical geometry, etc..

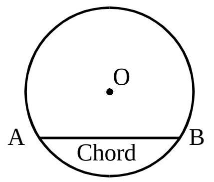
The concept of theoretical geometry also finds its application in other branches of Mathematics like Trigonometry, Calculus, etc.

In this chapter, we shall deal with circle, chord and tangent of a circle and similar triangles with respect to their properties and characters. This will enrich the learners with the basic concepts of circle and enable them to distinguish between chord and tangent and similar and congruent figures.

7.1 INTERNAL AND EXTERNAL INTERSECTION OF CHORDS OF A CIRCLE

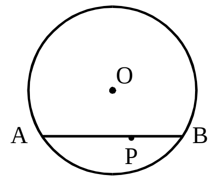
Review : In class IX, we have learnt about chord of a circle. In this section, we shall study theorems based on intersecting chords of a circle. Before this let us recall the definition for a chord of a circle.

Chord of a circle : A line segment joining any two points on a circle is called a chord of the circle.

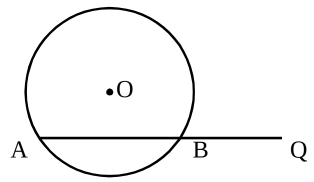


7.1.1 Segments of a chord

Internal divisor : Let AB be a chord of a circle. P be a point on the chord AB inside the circle. Then P is said to divide AB internally into two segments PA and PB.



External Divisor : If Q is a point on the chord AB produced, then Q is said to divide AB externally into two segments QA and QB.

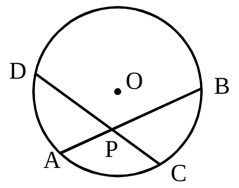


7.1.2 Theorem

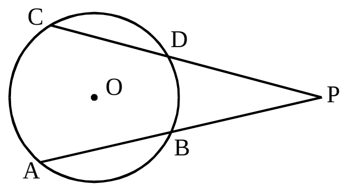
If two chords of a circle intersect inside the circle (or outside when produced), the rectangle formed by the two segments of one chord is equal in area to the rectangle formed by the two segments of the other chord. (Proof is not required).

Case (i) : Chords AB and CD intersect at P inside the circle then

$PA \times PB = PC \times PD$.



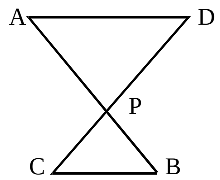
Case (ii) : Chords AB and CD when produced, intersect outside the circle at P then $PA \times PB = PC \times PD$.



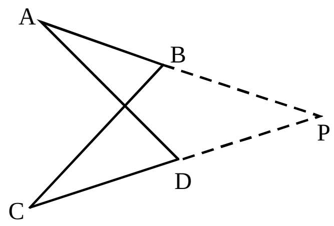
7.1.3 Theorem

Converse of theorem 7.1.2 :

If two line segments intersect internally (or externally, when produced) such that the area of the rectangle formed by the two parts of one is equal to the area of the rectangle formed by the two parts of the other, then the line segments are the two chords of a circle. (Proof is not required)



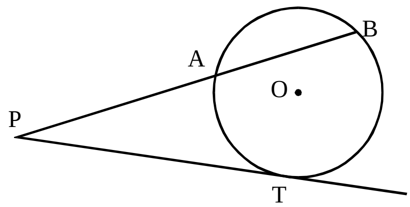
(i.e.,) Two line segments AB and CD intersect internally at P such that AP × PB = CP × PD then AB and CD are two chords of a circle.



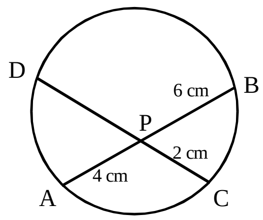
Two line segments AB and CD intersect externally (when produced) at P such that AP × PB = CP × PD then AB and CD are two chords of a circle.

7.1.4 Theorem

If PAB is a secant to a circle intersecting it at A and B and PT is a tangent at T, then PA × PB = PT^2. (Proof is not required)



Example 7.1 **:** Two chords AB and CD of a circle intersect each other (internally) at P. If AP = 4 cm, PB = 6 cm and CP = 2 cm, find PD.



Solution : By the given data the chords AB and CD intersect at P inside the circle.

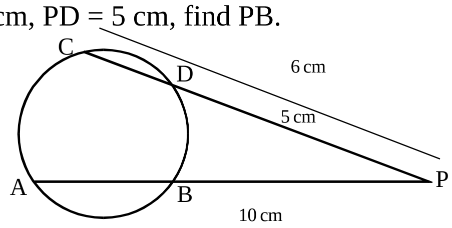
By theorem 7.1.2

AP × PB = CP × PD

4 × 6 = 2 × PD

∴ PD = 12 cm.

Example 7.2 **:** Two chords AB and CD of a circle intersect each other at P outside the circle. If AP = 10 cm, CP = 6 cm, PD = 5 cm, find PB.



Solution : Here the chords AB and CD intersect each other at P outside the circle.

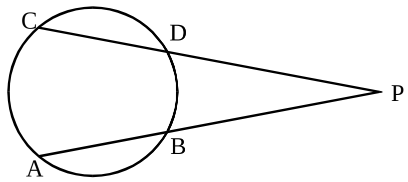
By theorem 7.1.2

$$PA \times PB = PC \times PD$$

$$10 \times PB = 6 \times 5 \Rightarrow PB = \frac{6 \times 5}{10} = 3 \text{ cm.}$$

Example 7.3 : AB and CD are two chords which when produced meet at P and if AP = CP. Show that AB = CD.

Solution : Given : AB and CD are two chords which when produced meet at P and AP = CP ... (1)



To prove : AB = CD

Proof : By the theorem 7.1.2

$$PA \times PB = PC \times PD$$

$$\Rightarrow \quad PA \times PB = PA \times PD \quad \text{(Since PA = PC)}$$

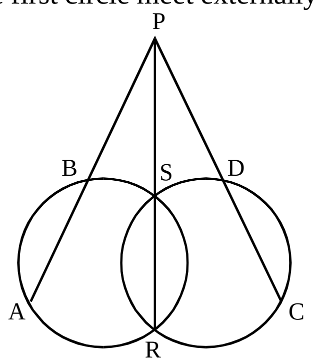
$$\Rightarrow \quad PB = PD \quad \ldots (2)$$

$$(1) - (2) \Rightarrow AP - PB = CP - PD \Rightarrow AB = CD$$

Hence proved.

Example 7.4 : In the figure P is a point on the common chord RS produced of two intersecting circles. AB and CD are the chords of the circles, they meet at P when produced. Prove that PA × PB = PC × PD.

Solution : Two chords AB and RS of the first circle meet externally at P.



By theorem 7.1.2

$PA \times PB = PR \times PS$ ... (1)

Two chords CD and RS of the second circle meet externally at P.

By theorem 7.1.2

$PC \times PD = PR \times PS$ ... (2)

From (1) and (2)

$PA \times PB = PC \times PD$

Example 7.5 : In the figure, AB is a diameter of a circle and PT is a tangent to the circle. If PB = 2 cm, PT = 8 cm, calculate the radius of the circle.

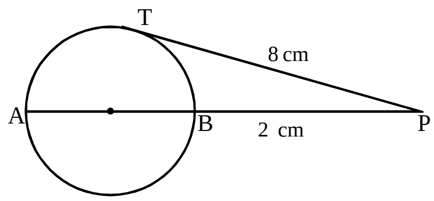
Solution : PT is a tangent and ABP is a secant to the circle.

$$\Rightarrow \quad PA \times PB = PT^2$$

$\Rightarrow \quad PA \times 2 = 8^2 \Rightarrow PA = 32$

But $PA = AB + BP$

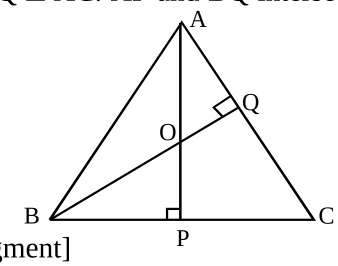
$\Rightarrow AB = 32 - 2 = 30$ cm



Hence, radius of the circle $= \frac{1}{2}$ (AB)

$= \frac{1}{2}$ (30) = 15 cm.

Example 7.6 : In Δ ABC, AP $\perp$ BC and BQ $\perp$ AC. AP and BQ intersect each other at O. Prove that AO × OP = BO × OQ.



Solution : $\angle APB = 90^o$ [since AP $\perp$ BC]

$\angle AQB = 90^o$ [since BQ $\perp$ AC]

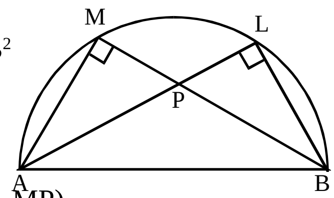
These two angles are subtended by the same line segment AB

and $\angle APB = \angle AQB$ [angles in the same segment]

Hence, the points B, P, Q, A are concyclic. (i.e.,) AP and BQ are the chords of a circle intersecting at O.

By theorem 7.1.2, AO × OP = BO × OQ

Example 7.7 : A semi circle is drawn on AB as a diameter and two chords AL and BM are drawn. They intersect each other at the point P. Prove that AB^2 = AL × AP + BM × BP.



Solution : From the figure $AB^2 = AM^2 + MB^2$

$= (AP^2 - MP^2) + MB^2$

$= AP^2 + MB^2 - MP^2$

$= AP^2 + (MB + MP)(MB - MP)$

$= AP^2 + (MB + MP) \times PB$

$= AP^2 + MB \times PB + MP \times PB$

$= AP^2 + MP \times PB + MB \times PB$

$= AP^2 + AP \times PL + MB \times PB$

[Since MP × PB = AP × PL]

$= AP \times (AP + PL) + MB \times PB$

$= AP \times AL + BP \times BM$

EXERCISE 7.1

1. AB and CD are two chords of a circle which intersect each other internally at P.

   (a) If CP = 3 cm, AP = 6 cm, PB = 2 cm, find PD

   (b) If AB = 11 cm, BP = 3 cm, CP = 4 cm, find CD.

   (c) If AP = 12 cm, AB = 15 cm, CP = PD fnd CD.

   (d) If AP = 5 cm, BP = 3 cm, CP = 2.5 cm, find DP.

2. AB and CD are two chords of a circle which intersect each other externally at P.

   (a) If CD = 2 cm, DP = 6 cm, BP = 3 cm, find AB.

   (b) If AB = 9 cm, BP = 3 cm, CP = 9 cm, find DP.

   (c) If AB = 5 cm, BP = 3 cm, PD = 2 cm, find CD.

3. AB and CD are two perpendicular chords of the circle. AD = 5 cm, AP = 3 cm, PB = 6 cm find CP where P is the point of intersection of AB and CD.

4. AB and CD are two chords of a circle intersecting at E. If E is the mid point of CD prove that AE × EB = CE^2.

5. ABT is a secant of a circle which intersects the circle at A and B and PT is a tangent to the circle at P,

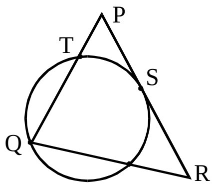
   (a) If PT = 5 cm, BT = 2.5 cm, find AB.

   (b) If PT = 12 cm, AB = 7 cm find BT.

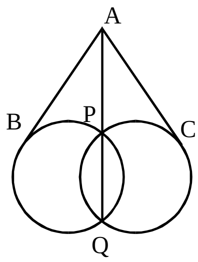
   (c) If AT = 9 cm, AB = 5 cm find PT.

6. In a trapezium ABCD, AB || CD and AD = BC. If P is the point of intersection of diagonals AC and BD, prove that PA × PC = PB × PD.

7. In fig. PQR is a triangle in which PQ = PR. A circle through Q touches PR at S and intersects PQ at T. If S is the mid point of PR, show that 4PT = PQ.



8. In the figure : Two circles intersect each other at points P and Q. If AB and AC are the tangents to the two circles from a point A on QP produced, show that AB = AC.



9. In a right triangle PQR, the perpendicular QT on the hypotenuse PR is drawn. Prove that (i) PR × PT = PQ^2, (ii) PR × TR = QR^2.

10. Two circles intersect each other at A and B. The common chord AB is produced to meet the common tangent PQ to the circles at C. Prove that CP = CQ.

7.2 COMMON TANGENT TO INTERSECTING AND NON-INTERSECTING CIRCLES

Review : In Class IX, we have learnt about tangent to a circle. In this section we shall study about tangent common to the two circles.

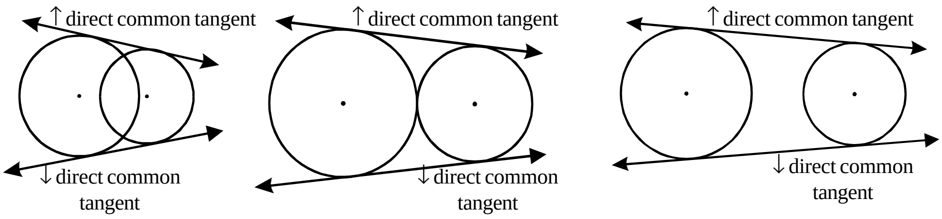
7.2.1 Common tangent to two circles

A line which touches two given circles is called a common tangent to the circle.

There are two types of common tangent. (i) Direct common tangent and (ii) Transverse common tangent.

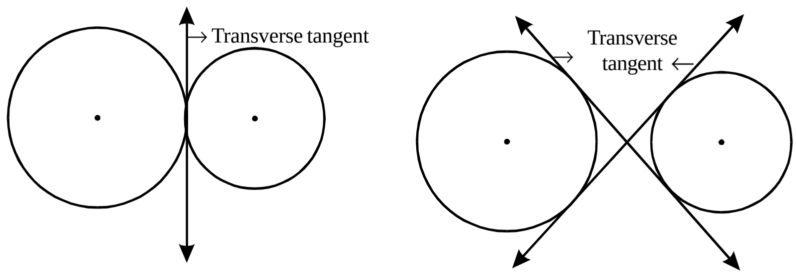
7.2.2 Direct common tangent

If a common tangent is drawn to two circles and if the common tangent lies on the same side of the centres of the circles, then it is called the direct common tangent.



7.2.3 Transverse common tangent

If the centres of the two circles lie on opposite side of the common tangent then the tangent is called the transverse common tangent.



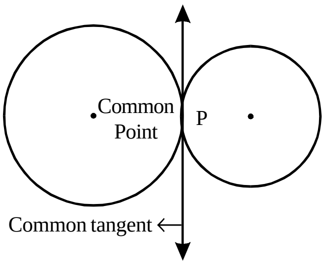
7.2.4 Intersecting circles

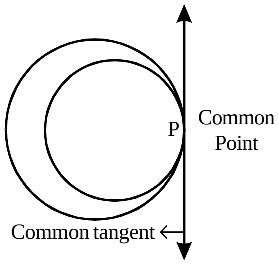
We know that two circles cannot intersect in more than two points. Therefore either (i) two circles intersect at two points (or) (ii) they intersect at one point (or) they do not intersect.

We shall discuss the circles intersecting in one point.

7.2.5 Two circles intersect in one point [(i.e.,) two circles touching each other]

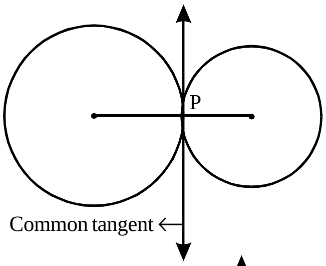
Two circles are said to touch each other if they meet at a point and can have a common tangent at this point.





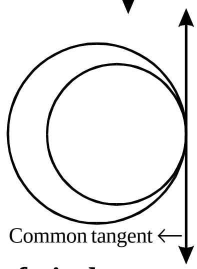
When two circles touch each other, either they touch each other internally or externally.

7.2.6 Two circles touching each other externally



Two circles are said to touch each other externally when they lie on the opposite sides of the common tangent.

7.2.7 Two circles touching each other internally

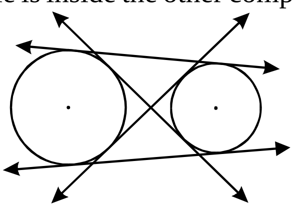


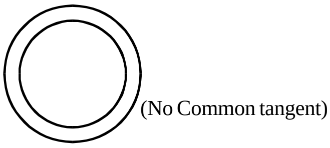
Two circles are said to touch each other internally when they lie on the same side of the common tangent.

7.2.8 Common tangents to different types of circles

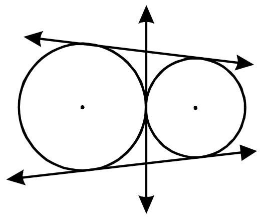
We shall discuss some other types of common tangents to two circles.

1. If two circles do not intersect, either two pairs of common tangents (one pair of direct common tangent and one pair of transverse common tangent) can be drawn or no common tangent can be drawn to the two circles (when one circle is inside the other completely).

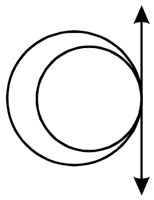




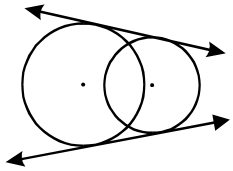
2. Let the two circles intersect in one point.



(i) If the circles touch externally, one pair of direct common tangents and only one transverse common tangent can be drawn to the two circles.



(ii) If two circles touch internally, only one direct common tangent can be drawn.



3. If two circles intersect in two points, one pair of direct common tangents can be drawn.

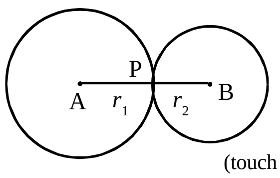
7.2.9 Theorem

If two circles touch each other (internally or externally) the point of contact lies on the line joining their centres. (Proof is not required)

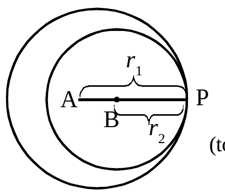
(i.e.,) C (A, r_1) and C(B, r_2) are the circles touching each other at a point P, then the points A, P and B are collinear.

Note : 1. When two circles touch each other externally the distance between the centres is equal to the sum of the radii. (i.e.,) AB = $r_1 + r_2$

Note : 2. When two circles touch each other internally, the distance between the cetres is equal to the difference of the radii. (i.e.,) AB = $r_1 - r_2$



(touch externally)



(touch internally)

Example 7.8 : Two circles of radii 4 cm and 6 cm touch each other. Find the distance between their centres when they touch (i) internally, (ii) externally.

Solution : Let the given radii of the circles be r_1 = 4 cm and r_2 = 6 cm.

(i) When the circles touch internally.

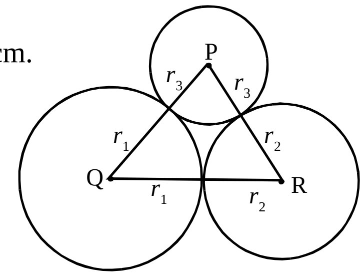
Distance between the centres $= r_2 - r_1 = 6 - 4 = 2$ cm

(ii) When the circles touch externally.

Distance between the centres $= r_1 + r_2 = 4 + 6 = 10$ cm.

Example 7.9 : With the vertices of a triangle PQR as centres, three circles are described, each touching the other two externally. If the sides of the triangle are 9 cm, 7 cm and 6 cm, find the radii of the circles.

Solution : By the given data



Let PQ = 9 cm, QR = 7 cm and RP = 6 cm.

$r_1 + r_2 = 7$ ... (1)

$r_2 + r_3 = 6$ ... (2)

$r_1 + r_3 = 9$ ... (3)

(1) + (2) + (3) we get

$2(r_1 + r_2 + r_3) = 22$

$r_1 + r_2 + r_3 = 11$ ... (4)

Subtracting (1) from (4) we get $r_3 = 4$

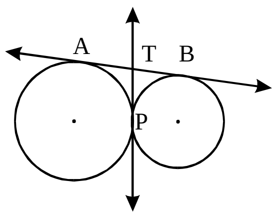
Subtracting (2) from (4) we get $r_1 = 5$

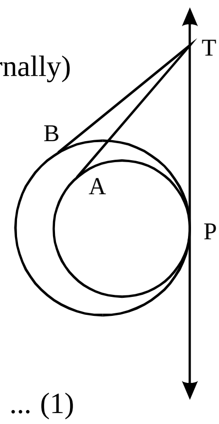
Subtracting (3) from (4) we get $r_2 = 2$

So, the radii of the circles are 5 cm, 2 cm and 4 cm.

Example 7.10 : Two circles touch externally or internally at a point P. From a point T on a common tangent PT, tangent segments TA and TB are drawn to the two circles. Prove that TA = TB.

Solution : Let the circles touch externally. (or internally)





We know that tangents from an external point to a circle are equal in length.

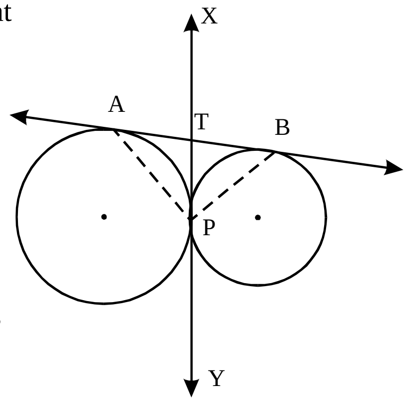
$\therefore$ TA = TP ... (1)

and TB = TP ... (2)

From (1) and (2) TA = TB

Example 7.11 : Two circles touch externally at a point P and a direct common tangent touch the circles at A and B. Prove that

(i) the common tangent at P bisects AB
(ii) AB subtends a right angle at P.



Solution : Let the point of intersection of the direct common tangent AB and the common tangent XY at P be T.

By the previous example. AT = TB

$\therefore$ The common tangent at P bisects AB

(ii) Now join AP and BP

In Δ ATP

TP = AT

$\Rightarrow \angle PAT = \angle APT$ ... (1)

In Δ PTB

TB = TP

$\Rightarrow \angle TBP = \angle TPB$ ...(2)

(1) + (2)

$$\angle PAT + \angle TBP = \angle APT + \angle TPB$$

$$\Rightarrow \angle PAT + \angle TBP = \angle APB$$

$$\Rightarrow \angle PAT + \angle TBP + \angle APB = \angle APB + \angle APB$$

$$\Rightarrow 180^o = 2\angle APB$$ [Since in Δ APB sum of the three angles = 180^o]

$$\Rightarrow \angle APB = 90^o$$

$\therefore$ AB subtends a right angle at P.

Example 7.12 : AB is a line segment and M is its mid point. Semi circles are drawn with AM, MB and AB as diameters on the same side of the line AB. A circle with centre O and radius r is drawn so that it touches all the three semi-circles.

Prove that $r = \frac{1}{6}$ AB.

Solution : Let AB = t units.

Let L, N be the mid points of AM, MB respectively.

Let C (0, r) be the given circle which touches the semi circles with centres L, M, N at P, R and Q respectively.

Now join OL and ON.

By theorem 7.2.9

The points O, P, L are collinear and the points O, Q, N are collinear.

By similarity points M, O, R are collinear.

Here AL = LM = MN = NB $= \dfrac{t}{4}$ units.

[since M is the mid point of AB, L is the mid point of AM and N is the mid point of MB]

Also OL = OP + LP

$= r + \dfrac{t}{4}$ [Since LP = $\dfrac{t}{4}$]

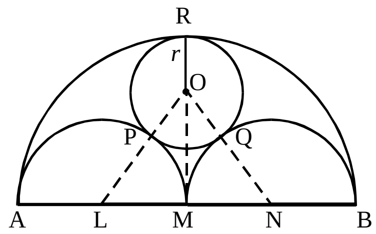
and ON = OQ + QN

$ON = r + \dfrac{t}{4}$

$\Rightarrow$ Δ OLN is an isosceles triangle and M is the mid point of the base LN.

$\Rightarrow$ OM $\perp$ LN

$\therefore$ OMN is a right angled triangle.



$\therefore$ $ON^2 = OM^2 + MN^2$

$$\left(r+\frac{t}{4}\right)^2 = (RM - r)^2 + \left(\frac{t}{4}\right)^2$$

$$\left(r+\frac{t}{4}\right)^2 = \left(\frac{t}{2}-r\right)^2 + \left(\frac{t}{4}\right)^2$$

$$\Rightarrow r = \frac{t}{6} \Rightarrow r = \frac{AB}{6}$$

EXERCISE 7.2

1. With the vertices of a triangle ABC as centres, three circles are described each touching the other two externally. If the sides of the triangle are 4 cm, 6 cm and 8 cm, find the radii of the circles.

2. The sides of Δ PQR are 8 cm, 10 cm and 12 cm. Three circles are drawn with centres P, Q and R each one touching the other two externally. Determine the radii of the circles.

3. Three circles of radii 3.5 cm, 4.5 cm and 5.5 cm touch each other externally. Find the length of sides of triangle formed by joining their centres.

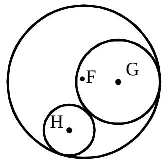
4. Find the distance between the centres of the circles when they touch (i) internally (ii) externally if the radii of the circles are (a) 5 cm and 7 cm, (b) 2.5 cm and 4.5 cm, (c) 8 cm and 10 cm.

5. Three circles with equal radii touch one another externally. Prove that the triangle formed by joining their centres is an equilateral triangle.

6. Figure represents three circles with centres F, G, H and radii 2 cm, 1.2 cm, 0.6 cm respectively, touching one another.

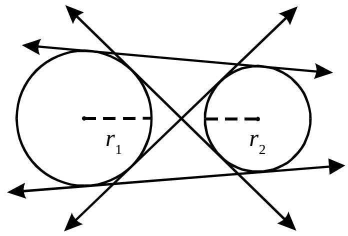
   (i) which of the contacts are internal ?

   (ii) Find the lengths of GH, HF and FG.



7. In figure, let radii of bigger and smaller circle be r_1 and r_2 respectively and d is the distance between their cenres then prove that (i) length of direct common tangent $= \sqrt{d^2 - (r_1 - r_2)^2}$

   (ii) length of transverse common tangent $= \sqrt{d^2 - (r_1 + r_2)^2}$



8. (i) Two circles of radii 16 cm and 9 cm touch each other externally. Find the length of the direct common tangent.

   (ii) Two circles of radii 25 cm and 9 cm touch each other externally. Find the length of the direct common tangent.

9. Two circles with radii a and b touch each other externally. Let c be the radius of the 3rd circle which touches these two circles as well as a common tangent to the two circles. Prove that are $\frac{1}{\sqrt{c}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}$ [where $a > b$]

7.3 BASIC PROPORTIONALITY THEOREM AND ITS CONVERSE

In this section, we will derive some results by using the basic proportionality theorem and solve the problems based on it.

7.3.1 Proportion

Let us take 4 line segments. If the ratio of the lengths of the first two line segments is equal to the ratio of the lengths of the other two line segments then the four line segments are said to be in proportion.

Let the ratio of the lengths of the first two line segments be $\frac{a}{b}$ and the ratio of the lengths of the other two segments be $\frac{c}{d}$. Then $\frac{a}{b} = \frac{c}{d}$.

7.3.2 Basic proportionality theorem (or) Thales theorem

If a straight line is drawn parallel to one side of a triangle it cuts the other two sides proportionally.

Given : A triangle ABC in which DE || BC and intersect AB in D and AC in E.

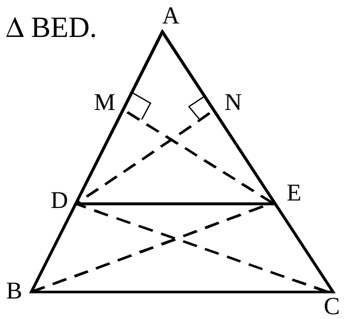
To prove : $\frac{AD}{DB} = \frac{AE}{EC}$

Construction : Join BE and CD.

Draw EM ⊥ BA and DN ⊥ CA.

Proof : In Δ ADE, EM ⊥ AD ⇒ EM is the height of Δ ADE.

In Δ BED, EM ⊥ AB ⇒ EM is the height of Δ BED.



Now

Area of triangle ADE = $\frac{1}{2}$ (Base × Height)

(i.e.,) area of Δ ADE = $\frac{1}{2}$ (AD.EM)

and Area of triangle BED = $\frac{1}{2}$ (Base × height)

(i.e.,) area of Δ BED = $\frac{1}{2}$ (DB × EM)

$$\therefore \quad \frac{\text{area of } \Delta ADE}{\text{area of } \Delta BED} = \frac{\frac{1}{2}(AD.EM)}{\frac{1}{2}(DB.EM)} = \frac{AD}{DB} \qquad \text{... (1)}$$

Similarly, $$\frac{\text{area of } \Delta ADE}{\text{area of } \Delta DCE} = \frac{\frac{1}{2}(AE.DN)}{\frac{1}{2}(EC.DN)} = \frac{AE}{EC} \qquad \text{... (2)}$$

But Δ BED and Δ DCE are on the same base DE and between the same parallels BC and DE.

∴ Area of Δ BED = Area of Δ DCE ... (3)

$$\therefore \qquad \frac{\text{area of } \Delta \text{ADE}}{\text{area of } \Delta \text{ BED}} = \frac{\text{area of } \Delta \text{ADE}}{\text{area of } \Delta \text{ DCE}} \qquad \text{[by using (1), (2) and (3)]}$$

$$\frac{AD}{DB} = \frac{AE}{EC}$$

Hence the theorem is proved.

The theorem is also known as Thales theorem.

Results : From the basic proportionality theorem,

We have $\qquad \frac{AD}{DB} = \frac{AE}{EC}$

Adding 1 on both sides, we get, $\frac{AD}{DB} + 1 = \frac{AE}{EC} + 1$

$$\frac{AD + DB}{DB} = \frac{AE + EC}{EC}$$

$$\frac{AB}{DB} = \frac{AC}{EC}$$

The above results can be summarised as follows :

(i) $\frac{AD}{DB} = \frac{AE}{EC}$ (ii) $\frac{DB}{AD} = \frac{EC}{AE}$ (iii) $\frac{AB}{AD} = \frac{AC}{AE}$

(iv) $\frac{AD}{AB} = \frac{AE}{AC}$ (v) $\frac{AB}{DB} = \frac{AC}{EC}$ (vi) $\frac{DB}{AB} = \frac{EC}{AC}$

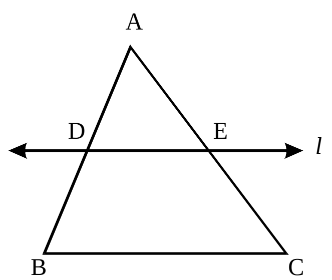
The converse of the ''basic proportionality'' theorem is also true.

7.4 CONVERSE OF BASIC PROPORTIONALITY THEOREM

If a line divides any two sides of a triangle in the same ratio, then the line is parallel to the third side. (Proof is not required)

(i.e.,) ABC is a triangle.

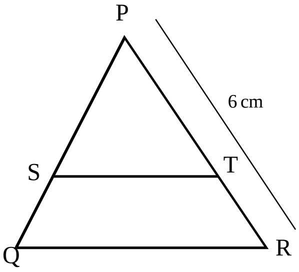
If a line '*l*' intersects side AB at D and side AC at E such that $\frac{AD}{DB} = \frac{AE}{EC}$ then *l* is parallel to BC (i.e.,) DE || BC.



Example 7.13 : In Δ PQR, ST || QR and $\frac{PS}{SQ} = \frac{3}{5}$ if PR = 6 cm find PT.

Solution : In Δ PQR, we have ST || QR.

By Basic proportionality theorem



$$\frac{PS}{SQ} = \frac{PT}{TR}$$

$$\Rightarrow \frac{PS}{SQ} = \frac{PT}{PR - PT}$$

$$\Rightarrow \frac{3}{5} = \frac{PT}{6 - PT}$$

$$\Rightarrow 3\,(6 - PT) = 5\,PT$$

$$\Rightarrow PT = \frac{9}{4} = 2.25 \text{ cm.}$$

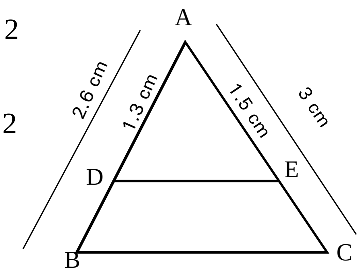
Example 7.14 : D and E are respectively the points on the sides AB and AC of a Δ ABC such that AB = 2.6 cm, AD = 1.3 cm, AC = 3 cm and AE = 1.5 cm, show that DE || BC.

Solution : Here

$$\frac{AB}{AD} = \frac{2.6}{1.3} = 2$$

$$\frac{AC}{AE} = \frac{3}{1.5} = 2$$

$$\frac{AB}{AD} = \frac{AC}{AE}$$



By the converse of BPT

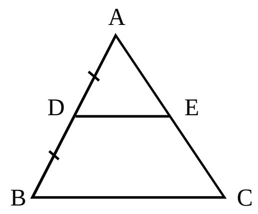
DE || BC.

Example 7.15 : The line drawn from the mid-point of one side of a triangle parallel to another side bisects the third side. Prove.

Solution : ABC be the given triangle. Let D be the mid point of side AB and DE is parallel to BC meeting AC at E.

To prove : E is the mid point of AC

In Δ ABC, we have DE || BC



By BPT, $\frac{AD}{DB} = \frac{AE}{EC}$

$$1 = \frac{AE}{EC} \quad \text{[since D is the mid point of AB} \Rightarrow \text{AD = DB]}$$

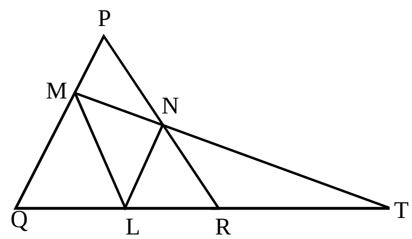
$\Rightarrow$ AE = EC. Hence E bisects AC.

Example 7.16 : L be a point on the side QR of Δ PQR. If LM, LN are drawn parallel to PR and QP meeting QP, PR at M, N respectively, MN meets produced QR in T prove that $LT^2 = RT \times QT$.

Solution : In Δ MLT

We have NR || ML

By BPT, $\frac{TR}{TL} = \frac{TN}{TM}$ ... (1)



In Δ MQT

We have LN || QM

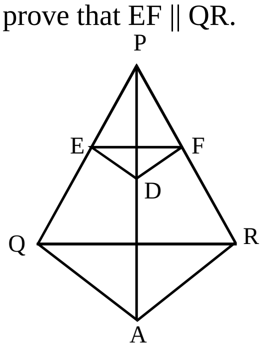
By BPT, $\frac{TL}{TQ} = \frac{TN}{TM}$ ... (2)

From (1) and (2) $\frac{TR}{TL} = \frac{TL}{TQ} \Rightarrow TR \times TQ = TL^2$

$$\therefore \; TL^2 = TR \times TQ.$$

Example 7.17 : In fig, DE || AQ and DF || AR prove that EF || QR.

Solution : In Δ PQA, ED || QA



By BPT, $\frac{PE}{EQ} = \frac{PD}{DA}$ ... (1)

In Δ PAR, DF || AR,

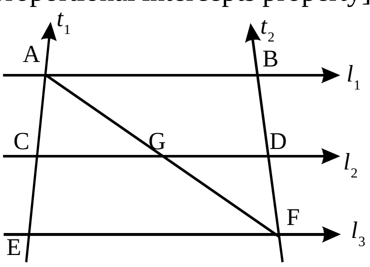
$$\frac{PD}{DA} = \frac{PF}{FR} \text{ ... (2)}$$

From (1) and (2), $\frac{PE}{EQ} = \frac{PF}{FR}$

By the converse of BPT, EF || QR.

Example 7.18 : If three or more parallel lines are intersected by two transversals, prove that the intercepts made by them on the transversal are proportional.

[This result is generally referred to as the proportional intercepts property]



Solution : Straight lines l_1, l_2 and l_3 are parallel to each other.

These three parallel lines are cut by the transversals t_1 and t_2 at the points. A, C, E and B, D, F respectively.

Now join AF, which meets CD at G.

In Δ AEF, CG || EF

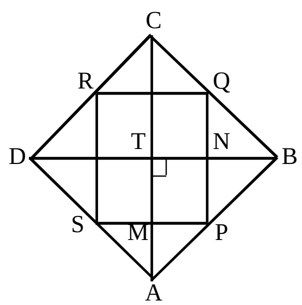
By BPT, $\frac{AC}{CE} = \frac{AG}{GF}$ ... (1)

In Δ AFB, GD || AB

By BPT $\frac{GA}{FG} = \frac{DB}{FD}$ ... (2)

From (1) and (2) $\frac{AC}{CE} = \frac{BD}{DF}$

Example 7.19 : Prove that the line segment joining the mid points of the adjacent sides of a rhombus form a rectangle.



Solution : Let ABCD be a rhombus and P, Q, R, S be the mid points of the sides AB, BC, CD and DA respectively.

Now join PQ, QR, RS and SP.

Now to prove : PQRS is a rectangle.

Now join DB, In Δ ADB,

$\frac{AS}{SD} = 1$ and $\frac{AP}{PB} = 1$ [Since S and P are the mid points of AD and AB respectively]

$\Rightarrow \frac{AS}{SD} = \frac{AP}{PB}$

By the converse of BPT SP || DB.

Similarly RQ || DB $\Rightarrow$ SP || RQ

Now join AC, similarly we can prove PQ || SR

$\therefore$ PQRS is a parallelogram.

In this figure PNTM is also a parallelogram.

Here $\angle$ MTN = 90^o [since Diagonals of a rhombus are $\perp$]

$\Rightarrow$ $\angle$ MPN = 90^o [since opposite angles of a parallelogram]

Similarly $\angle$PQR = $\angle$QRS = $\angle$RSP = 90^o

$\therefore$ PQRS is a rectangle.

EXERCISE 7.3

1. In a Δ ABC, D and E are points on the sides AB and AC respectively such that DE || BC.

   (a) If $\frac{AD}{DB} = \frac{3}{2}$ and AC = 15 cm, find AE.

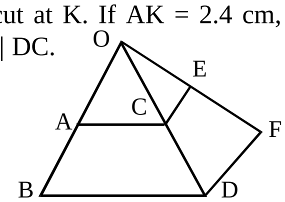
   (b) If AD = 4 cm, AE = 8 cm, DB = (x – 4) cm and EC = 3x – 19 find x.

   (c) If AD = 3 cm, DB = 2 cm, AE = 2.7 cm find AC.

2. If D and E are respectively the points on the sides AB and AC of a triangle ABC such that AD = 3 cm, DB = 4.5 cm, AE = 4 cm and EC = 6 cm then show that DE || BC.

3. The diagonals of a quadrilateral ABCD cut at K. If AK = 2.4 cm, KC = 1.6 cm, BK = 1.5 cm, KD = 1 cm, prove that AB || DC.

4. In fig, AC || BD and CE || DF.

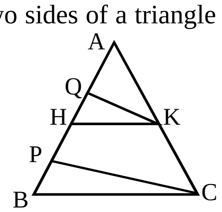


   (a) If OA = 10.5 cm, AB = 4.5 cm, OD = 7 cm, find CD.

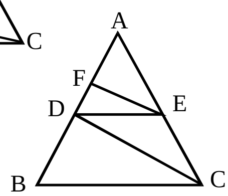
   (b) If OA = 12 cm, AB = 9 cm, OC = 8 cm, EF = 4.5 cm find CD and OF.

5. Prove that the line joining the mid points of two sides of a triangle is parallel to the third side.

6. In fig, if AQ = 6 cm, QH = 4 cm, HP = 5 cm, KC = 18 cm, find AK and PB. [Here PC || QK and BC || HK]



7. In the given fig. if DE || BC and CD || EF, prove that AD^2 = AB × AF.



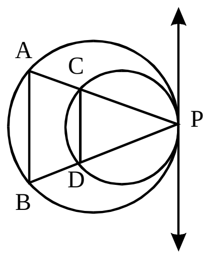
8. Prove that the line segments joining the mid-points of the adjacent sides of a quadrilateral form a parallelogram.

9. Prove that the diagonals of a trapezium divide each other proportionally. [Hint : From the point of intersection of the diagonals, draw the line parallel to parallel sides to meet one of the non-parallel sides in a point.]

10. ABCD is a quadrilateral. From any point P on AB a line drawn parallel to BC meets AC at Q and the line through Q drawn parallel to CD meets AD at R. Prove that PR || BD.

11. The side BC of a triangle ABC is bisected at D. O is any point in AD. BO and CO produced meet AC and AB in E and F respectively and AD is produced to G so that D is the mid point of OG. Prove that AO : AG = AF : AB and show that FE || BC.

12. Two circles touch each other internally at P. If ACP and BDP are the lines meeting the two circles at A, B and C, D respectively. Prove that $\frac{AP}{CP} = \frac{BP}{DP}$



13. In Δ ABC, ∠C is a right angle. P is a point on AB. PN ⊥ CB. If AP = 3, PB = 4, CN = x, PN = y show that $y = \frac{4}{3}\sqrt{9 - x^2}$

7.5 ANGLE BISECTOR THEOREM

In this section, we will prove angle bisector theorem by using the basic proportionality theorem.

7.5.1 Angle Bisector Theorem

If the vertical angle of a triangle is bisected internally or externally, the bisector divides the base internally or externally into two segments which have the same ratio as the order of two sides of the triangle.

Given : Case (i) : Δ ABC in which AD is the internal bisector of ∠A which meets BC at D.

To prove : $\frac{BD}{DC} = \frac{AB}{AC}$

Construction : Draw CE || DA to meet BA produced at E.

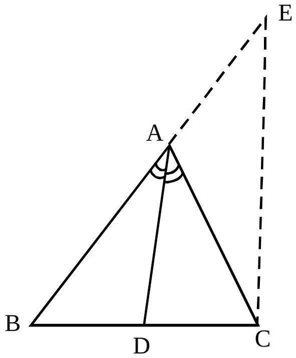
Proof : Since DA || CE,

$\Rightarrow \angle DAC = \angle ACE$ ... (1)

[since alternate angles]

and $\angle BAD = \angle AEC$ ... (2)

[corresponding angles]



Since AD is the angular bisector of $\angle A$.

$\angle BAD = \angle DAC$ ... (3)

From (1), (2) and (3)

In Δ ACE, $\angle ACE = \angle AEC$

$\Rightarrow$ AE = AC ... (4) [since Sides opposite to equal angles are equal]

Now in Δ BCE, we have, DA || CE

By BPT, $\frac{BD}{DC} = \frac{BA}{AE}$

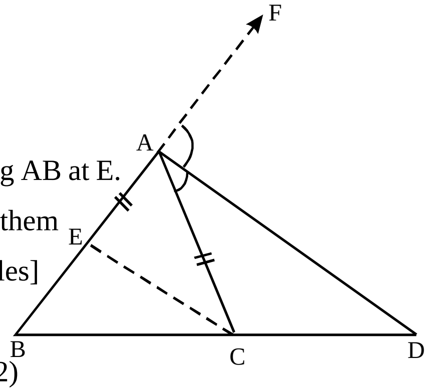
$\frac{BD}{DC} = \frac{AB}{AC}$ from (4)

Hence proved.

Case (ii) : Δ ABC, in which AD is the bisector of the exterior $\angle A$ and intersects BC produced at D.

To prove : $\frac{BD}{DC} = \frac{AB}{AC}$

Construction : Draw CE || DA meeting AB at E.



Proof : Since CE || DA and AC cuts them

$\angle ECA = \angle CAD$ ...(1) [alternate angles]

Also CE || DA and BF cuts them

$\angle CEA = \angle DAF$ ... (2)

But AD is the bisector of $\angle CAF \Rightarrow \angle CAD = \angle DAF$ ...(3)

From (1), (2) and (3) $\angle CEA = \angle ECA$

In Δ ECA, $\angle CEA = \angle ECA$

$\Rightarrow$ AC = AE ... (4) [since sides opposite to equal angles are equal]

In Δ BDA, EC || AD

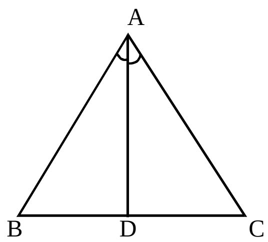
By BPT, $\frac{BD}{DC} = \frac{BA}{AE}$

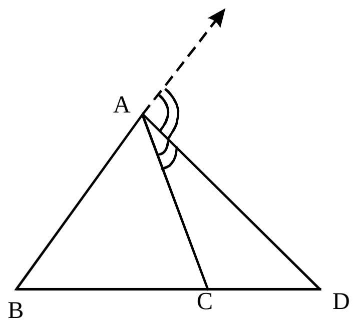
$\Rightarrow \frac{BD}{DC} = \frac{AB}{AC}$ from (4)

Hence proved.

7.6 CONVERSE OF ANGLE BISECTOR THEOREM

If a side of a triangle is divided internally or externally which have the same ratio as the other two sides, the straight line joining the point of intersection to the opposite vertex bisects the vertical angle internally or externally. (Proof is not required)





(i.e.,) In a triangle ABC, if D is a point on BC

such that $\frac{BD}{DC} = \frac{AB}{AC}$ then AD is a bisector of $\angle$A.

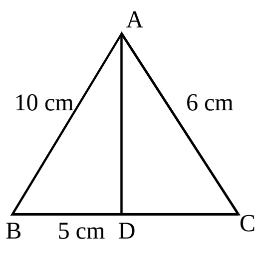
Example 7.20 : In Δ ABC, AD is the bisector of $\angle$A if AB = 10 cm, AC = 6 cm, BD = 5 cm find DC.

Solution : Δ ABC, AD is the bisector of $\angle$A.

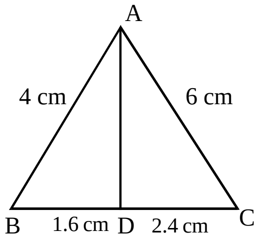
$\Rightarrow \frac{BD}{DC} = \frac{AB}{AC}$

$\Rightarrow \frac{5}{DC} = \frac{10}{6}$

$\therefore$ DC = 3 cm.



Example 7.21 : Check whether AD is the bisector of ∠A of Δ ABC in the figure given. AB = 4 cm, AC = 6 cm, BD = 1.6 cm and CD = 2.4 cm.



Solution : Here, $\dfrac{BD}{DC} = \dfrac{1.6}{2.4} = \dfrac{2}{3}$ ... (1)

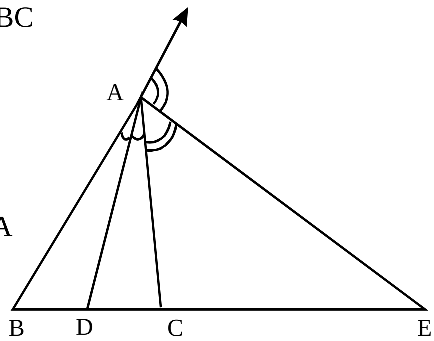
$\dfrac{AB}{AC} = \dfrac{4}{6} = \dfrac{2}{3}$ ... (2)

From (1) and (2) $\dfrac{BD}{DC} = \dfrac{AB}{AC}$

By the converse of ABT, AD is the bisector of ∠A.

Example 7.22 : The bisector of interior ∠A of Δ ABC meets BC at D, and the bisector of exterior ∠A meets BC produced at E. Prove that $\dfrac{BD}{BE} = \dfrac{CD}{CE}$.

Solution : By the given data, in Δ ABC



AD is the internal bisector of ∠A

$\dfrac{BD}{DC} = \dfrac{AB}{AC}$ ... (1)

and AE is the external bisector of ∠A

$\dfrac{BE}{EC} = \dfrac{AB}{AC}$ ... (2)

From (1) and (2) $\Rightarrow \dfrac{BD}{DC} = \dfrac{BE}{EC}$

$\Rightarrow \dfrac{BD}{BE} = \dfrac{CD}{CE}$.

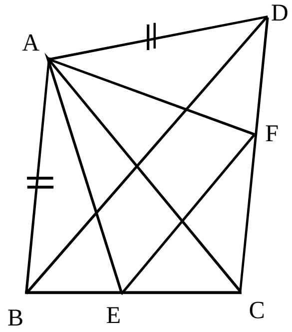
Hence proved.

Example 7.23 : ABCD is a quadrilateral with AB = AD. AE and AF are bisectors of ∠BAC and ∠DAC respectively. Prove that EF || BD.

Solution : Given AB = AD

AE is the bisector of $\angle BAC$ ... (1)

AF is the bisector of $\angle DAC$ ... (2)



Now join BD and EF

From (1)

$$\frac{BE}{EC} = \frac{AB}{AC} \quad ... (3)$$

From (2)

$$\frac{DF}{FC} = \frac{AD}{AC}$$

$$\frac{DF}{FC} = \frac{AB}{AC} \quad ... (4) \quad \text{(since AD = AB)}$$

From (3) and (4) $\frac{BE}{EC} = \frac{DF}{FC} \Rightarrow \frac{CE}{EB} = \frac{CF}{FD}$

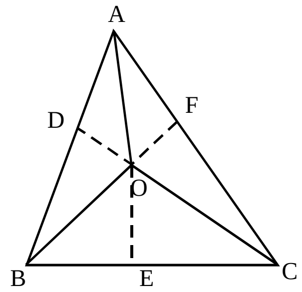
In Δ BCD, $\frac{CE}{EB} = \frac{CF}{FD}$

By the converse of BPT, EF || BD. Hence proved.

Example 7.24 : O is the point inside a triangle ABC. The bisector of $\angle AOB$, $\angle BOC$, $\angle COA$ meet the sides AB, BC and CA at the points D, E, F respectively. Prove that AD.BE.CF = DB.EC.FA.

Solution : By the given data OE is the bisector of $\angle BOC$

$$\therefore \frac{BE}{EC} = \frac{BO}{OC} \quad ... (1)$$



Also, OF is the bisector of $\angle COA$

$$\therefore \frac{CF}{FA} = \frac{CO}{OA} \quad ... (2)$$

OD is the bisector of $\angle AOB$.

$$\therefore \frac{AD}{DB} = \frac{AO}{OB} \quad ... (3)$$

$$(1) \times (2) \times (3) \Rightarrow \frac{BE}{EC} \cdot \frac{CF}{FA} \cdot \frac{AD}{DB} = \frac{BO}{OC} \cdot \frac{CO}{OA} \cdot \frac{AO}{OB}$$

$$\frac{BE}{EC} \cdot \frac{CF}{FA} \cdot \frac{AD}{DB} = 1$$

$$AD.BE.CF = DB.EC.FA$$

Hence proved.

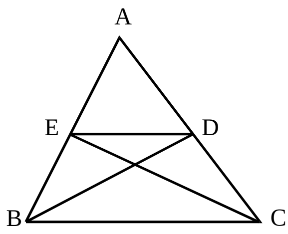
Example 7.25 : The bisector of the angle B and C of a triangle ABC, meet the opposite sides in D and E respectively, if DE || BC, prove that the triangle is isosceles.

Solution : By the given data

BD is the angular bisector of $\angle B \Rightarrow \frac{AD}{DC} = \frac{AB}{BC}$ ... (1)

and CE is the angular bisector of $\angle C$

$$\Rightarrow \frac{AE}{EB} = \frac{AC}{CB} \quad ... (2)$$



and DE || BC.

By BPT, $\frac{AE}{EB} = \frac{AD}{DC}$ ... (3)

From (1), (2) and (3) $\frac{AB}{BC} = \frac{AC}{BC} \Rightarrow AB = AC$

Hence Δ ABC is isosceles.

EXERCISE 7.4

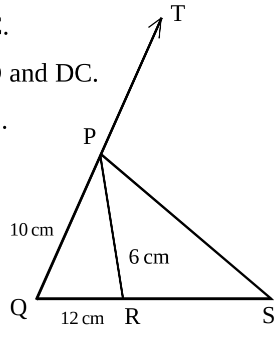
1. In a Δ ABC, AD is the bisector of $\angle A$, meeting side BC at D.

   (a) If BD = 2.5 cm, AB = 5 cm and AC = 4 cm find DC.

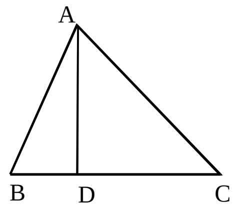
   (b) If AB = 10 cm, AC = 14 cm and BC = 6 cm find BD and DC.

   (c) If AB = 3.6 cm, BC = 4.5 cm, BD = 2.7 cm, find AC.

2. In fig. PS is the bisector of the exterior $\angle RPT$ meeting QR produced at S. If PQ = 10 cm, PR = 6 cm, and QR = 12 cm find RS.



3. Check whether in the following AD is the bisector of $\angle A$ of Δ ABC given in figure.

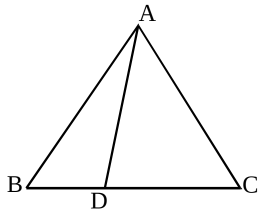


(a) AB = 8 cm, AC = 24 cm, BD = 6 cm and BC = 24 cm

(b) AB = 5 cm, AC = 10 cm, BD = 2.5 cm and CD = 3.5 cm.

4. In figure, $\frac{AB}{AC} = \frac{BD}{DC}$ and

B = $\angle 40^o$, C = $\angle 60^o$ find $\angle BAD$.



5. If the bisector of an angle of a triangle bisects the opposite side, prove that the triangle is isosceles.

6. The diagonal BD of a quadrilateral ABCD bisects $\angle B$ and $\angle D$ prove that $\frac{AB}{BC} = \frac{DA}{CD}$.

7. A, B, C are points on sides QR, RP and PQ respectively of Δ PQR such that PA bisects $\angle P$ QB bisects $\angle Q$ and RC bisects $\angle R$. If PQ = 6 cm, QR = 8 cm and RP = 4 cm, determine PC, QA, RB.

8. D is the midpoint of side BC of Δ ABC. DP bisects $\angle ADB$ meeting AB at P and DQ bisects $\angle ADC$ meeting AC at Q. Prove that PQ || BC.

9. ABCD is a quadrilateral. The bisectors of $\angle A$ and $\angle C$ meet on BD. Prove that the bisectors of $\angle B$ and $\angle D$ meet on AC.

10. Prove that the internal bisectors of the angle of a triangle are concurrent.

7.7 SIMILAR TRIANGLES

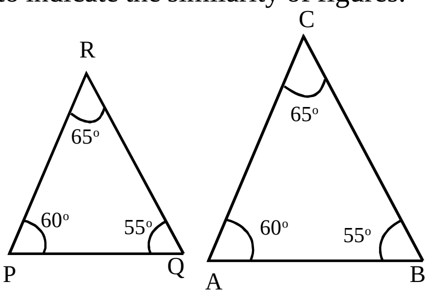
7.7.1 Similar Figures

Two figures are called similar if and only if they have same shape, not necessarily the same size.

We use symbol ||| (read as similar to) to indicate the similarity of figures.

7.7.2 Equiangular

Two polygons are said to be equiangular to one another if the angles of the first polygon, taken in order, are respectively equal to the angles of the second polygon taken in the same order.



$\therefore$ Δ PQR and ΔABC are equiangular.

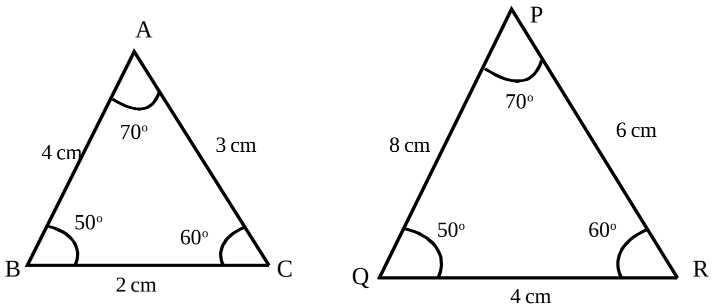
7.7.3 Similar Polygon

Two polygons are said to be similar to one another if

(i) the polygons are equiangular to one another

(ii) the ratio of any side of the first polygon to the corresponding side of the second is the same. [(i.e.,) their corresponding sides are proportional]

Example :



From the figure,

$$\angle A = \angle P, \quad \angle B = \angle Q, \quad \angle C = \angle R$$

and $$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR} \quad = \frac{4}{8} = \frac{2}{4} = \frac{3}{6}$$

$$\therefore \Delta ABC \parallel\!| \Delta PQR.$$

Note (i) : Polygons which are equiangular to one another need not be similar.

For example, consider a square and any rectangle. They are equiangular to one another but they are not similar because their corresponding sides are not proportional.

Note (ii) : Polygons in which corresponding sides are proportional need not be similar.

For example Consider a square and a rhombus their corresponding sides are proportional but they are not similar because the square and rhombus are not equiangular to one another.

(i..e.,) For similarity both conditions. [Corresponding angles are equal and corresponding sides are proportional] should be satisfied.

But we will see about that, later.

Triangles are special type of polygons in which if one condition of similarity holds, the other is true automatically. (i.e.,) (i) if two triangles are equiangular, then their corresponding sides are proportional (ii) if the corresponding sides are proportional, then they are equiangular also.

Notation : Similar triangles and polygons should always be named so that the order of the letters indicates the correspondence between the two figures.

(i.e.,) Δ ABC ||| Δ XYZ means that $\angle A = \angle X$, $\angle B = \angle Y$, $\angle C = \angle Z$

$$\text{and } \frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX}$$

Note : Two congruent figures are always similar, but the converse may not be true. (i.e.,) two similar figures may not be congruent.

7.7.4 Theorem : [AAA Similarity]

If two triangles are equiangular to one another, then the two triangles are similar.

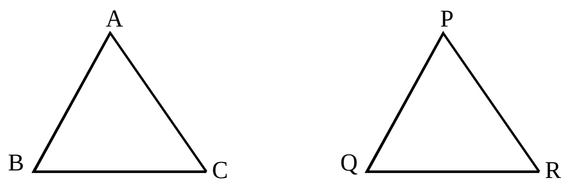
Given : Two triangles ABC and PQR such that $\angle A=\angle P$, $\angle B=\angle Q$, $\angle C=\angle R$.

To prove : Δ ABC ||| Δ PQR

(i.e.,) **To prove :** $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$

Proof :

Case (i) : When AB = PQ.



In this case we have

$\angle A = \angle P$, $\angle B = \angle Q$, $\angle C = \angle R$ and AB = PQ.

$\therefore$ by ASA congruent rule

$\Delta ABC \equiv \Delta PQR$

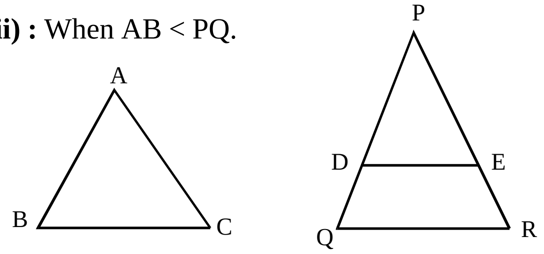
Corresponding sides are equal

AB = PQ, BC = QR, AC = PR.

$$\Rightarrow \frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$$

By the definition of similarity $\Delta ABC \,|||\, \Delta PQR$.

Case (ii) : When AB < PQ.



Mark a point D on PQ and E on PR such that AB = PD and AC = PE. Join DE. In triangles ABC and PDE.

AB = PD, $\angle A = \angle P$ and AC = PE.

By SAS criterion of congruence.

$\Delta ABC \equiv \Delta PDE$

$\Rightarrow \quad \angle B = \angle PDE$

But $\quad \angle B = \angle Q$

$\Rightarrow \quad \angle PDE = \angle PQR$

$\Rightarrow \quad DE \parallel QR$ (since Corresponding angles are equal)

$$\Rightarrow \quad \frac{PD}{PQ} = \frac{PE}{PR} \quad \text{(By BPT)}$$

$$\Rightarrow \quad \frac{AB}{PQ} = \frac{AC}{PR}$$

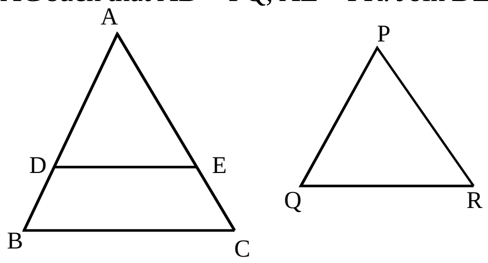
Similarly, we can prove that, $\frac{AB}{PQ} = \frac{BC}{QR}$

$$\therefore \frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$$

Hence $\Delta ABC \,|||\, \Delta PQR$.

Case (iii) : When AB > PQ.

Mark a point D on AB and E on AC such that AD = PQ, AE = PR. Join DE.



In Δ ADE and Δ PQR

AD = PQ

∠A = ∠P

AE = PR

By SAS criterion of congruence

Δ ADE ≡ Δ PQR

⇒ ∠ADE = ∠Q But ∠B = ∠Q

⇒ ∠ADE = ∠B

⇒ DE || BC [Corresponding angles are equal]

⇒ $\frac{AB}{AD} = \frac{AC}{AE}$ (By BPT)

⇒ $\frac{AB}{PQ} = \frac{AC}{PR}$

Similarly we can prove $\frac{AB}{PQ} = \frac{BC}{QR}$

∴ $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP}$

∴ Δ ABC ||| Δ PQR

Remark : From the above theorem we conclude that two triangles are similar iff they are equiangular.

Corollary : (AA similarity)

If two angles of one triangle are respectively equal to two angles of another triangle, then the two triangles are similar.

Proof : Let Δ ABC and Δ PQR be two triangles such that ∠A = ∠P, ∠B = ∠Q.

In Δ ABC and Δ PQR we have

∠A + ∠B + ∠C = 180, ∠P + ∠Q + ∠R = 180°

$\Rightarrow \quad \angle A + \angle B + \angle C = \angle P + \angle Q + \angle R$

$\Rightarrow \quad \angle C = \angle R$

$\angle A = \angle P, \angle B = \angle Q, \angle C = \angle R$

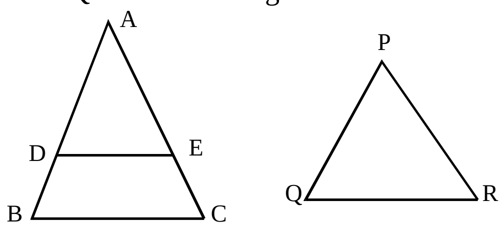
Δ ABC ||| Δ PQR (AAA similarity theorem)

7.8 Theorem [SAS Similarity]

If two triangles have one angle of the one equal to one angle of the other and the sides about the equal angles proportional, then the two triangles are similar.

Given : ABC and PQR are two triangles such that $\angle A = \angle P$ and

$\frac{AB}{PQ} = \frac{AC}{PR}$.



Construction : Mark a point D and E on AB and AC respectively such that AD = PQ and AE = PR. Join DE.

Proof : In Δ ADE and Δ PQR, AD = PQ, $\angle A = \angle P$ and AE = PR

Therefore by SAS criterion of congruence, we have

$\Delta ADE \equiv \Delta PQR. \Rightarrow \angle ADE = \angle PQR ; \angle AED = \angle PRQ$ ... (1) (c.p. c.t)

Now $\frac{AB}{PQ} = \frac{AC}{PR}$ (given condition)

$\Rightarrow \quad \frac{AB}{AD} = \frac{AC}{AE}$

$\Rightarrow$ DE || BC (By the converse of BPT)

$\Rightarrow \quad \angle ADE = \angle B$ and $\angle AED = \angle C$ ... (2)

From (1) and (2), $\angle PQR = \angle B$ and $\angle PRQ = \angle C$

Now $\angle A = \angle P, \angle B = \angle Q, \angle C = \angle R$

By AA similarity, Δ ABC ||| Δ PQR.

Remark : If two triangles are similar, then their corresponding sides are proportional and they are proportional to the corresponding perimeters.

Proof : Let Δ ABC and Δ PQR be similar

then $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$

$\Rightarrow \quad \frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR} = \frac{AB+BC+AC}{PQ+QR+PR}$ (using ratio and proportion)

$= \frac{\text{Perimeter of } \Delta ABC}{\text{Perimeter of } \Delta PQR}$

7.9 Theorem : [SSS Similarity]

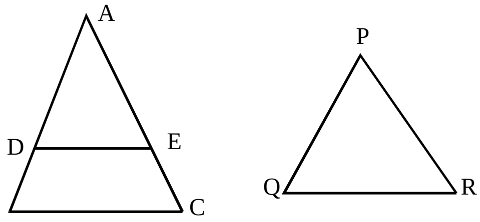
If two triangles have their corresponding sides proportional then the two triangles are similar.

Given : Two triangles ABC and PQR such that

$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$$

To prove : Δ ABC ||| Δ PQR.

Construction : Mark a point D and E on AB and AC respectively such that AD = PQ and AE = PR. Join DE.



Proof : We have $\frac{AB}{PQ} = \frac{AC}{PR}$

$\Rightarrow \frac{AB}{AD} = \frac{AC}{AE}$ [since AD = PQ, AE = PR]

$\Rightarrow$ DE || BC [since By the converse of BPT]

$\Rightarrow \angle ADE = \angle B, \angle AED = \angle C$ ... (I)

[since Corresponding angles of parallel lines]

In Δ ADE and Δ ABC. ∠A is common and $\frac{AB}{AD} = \frac{AC}{AE}$

$\Rightarrow$ Δ ABC ||| Δ ADE (By SAS criterion of similarity)

$$\Rightarrow \qquad \frac{AB}{AD} = \frac{BC}{DE} = \frac{AC}{AE} \qquad \text{... (1)}$$

$$\text{But} \qquad \frac{AB}{AD} = \frac{AB}{PQ} = \frac{BC}{QR} \qquad \text{... (2)}$$

From (1) and (2), $\frac{BC}{DE} = \frac{BC}{QR}$

(i.e.,) DE = QR

$\therefore$ In Δ ADE and Δ PQR, AD = PQ, AE = PR, DE = QR

$\therefore$ Δ ADE $\equiv$ Δ PQR. (By SSS congruency)

$\therefore$ $\angle A = \angle P$, $\angle ADE = \angle PQR$, $\angle AED = \angle PRQ$ ... (II)

From (I) and (II), $\angle A = \angle P$

$\angle B = \angle Q$

$\angle C = \angle R$

By AAA criterion for similarity, Δ ABC ||| Δ PQR.

7.10 Theorem :

Similar triangles are to one another as the squares on their corresponding sides.

(OR)

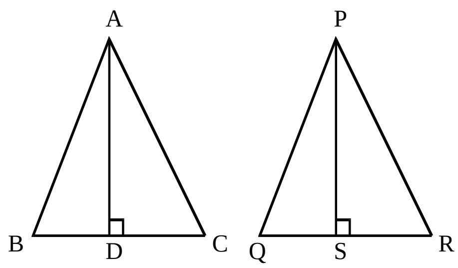
The ratio of the areas of two similar triangles is equal to the ratio of the squares of their corresponding sides.

Given : Δ ABC and Δ PQR such that Δ ABC ||| Δ PQR.

To prove : $\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta PQR} = \frac{AB^2}{PQ^2} = \frac{BC^2}{QR^2} = \frac{CA^2}{RP^2}$

Construction :

Draw AD $\perp$ BC and PS $\perp$ QR.



Proof : $\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta PQR} = \frac{\frac{1}{2} \times BC \times AD}{\frac{1}{2} \times QR \times PS}$

$$= \frac{BC}{QR} \times \frac{AD}{PS} \quad \ldots (1)$$

Now in triangles ADB and PSQ.

$\angle B = \angle Q$ [since Δ ABC ||| Δ PQR]

$\angle ADB = \angle PSQ$ [since Each 90^o]

Therefore by AA similarity, Δ ADB ||| Δ PSQ

$$\Rightarrow \quad \frac{AD}{PS} = \frac{AB}{PQ} \quad \ldots (2)$$

But $\frac{AB}{PQ} = \frac{BC}{QR}$

$$\frac{AD}{PS} = \frac{BC}{QR} \quad \ldots (3) \quad \text{[from (2)]}$$

Therefore $\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta PQR} = \frac{BC}{QR} \times \frac{BC}{QR}$ [From (1) and (2)]

$$= \frac{BC^2}{QR^2} \quad \ldots (4)$$

As Δ ABC ||| Δ PQR, $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP}$ ... (5)

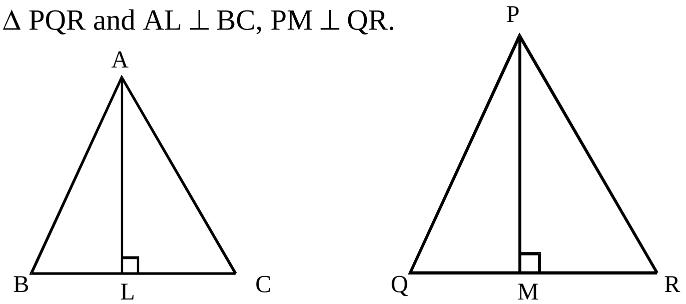
Hence $\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta PQR} = \frac{AB^2}{PQ^2} = \frac{BC^2}{QR^2} = \frac{CA^2}{RP^2}$ [From (4) and (5)]

Results (1) :

The ratio of the areas of two similar triangles is equal to the ratio of the squares of their corresponding altitudes.

Given : Let Δ ABC and Δ PQR be the given triangles such that

Δ ABC ||| Δ PQR and AL $\perp$ BC, PM $\perp$ QR.



To prove : $\dfrac{\text{Area of } \Delta ABC}{\text{Area of } \Delta PQR} = \dfrac{AL^2}{PM^2}$

Proof : We know that, the ratio of the areas of two similar triangles is equal to the ratio of the squares of any pair of their corresponding sides.

$$\therefore \frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta PQR} = \frac{AB^2}{PQ^2} \qquad \text{... (1)}$$

Now consider the triangles ABL and PQM

Here $\quad \angle ALB = \angle PMQ = 90^o$

$\angle B = \angle Q \qquad (\because \Delta ABC ||| \Delta PQR)$

By AA similarity, $\Delta ALB ||| \Delta PMQ$

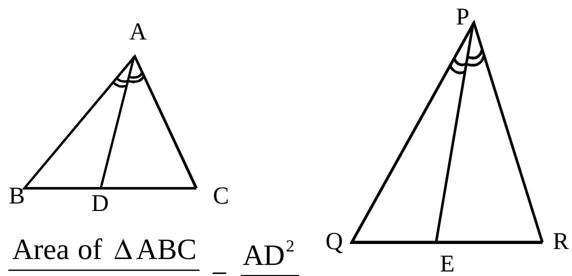
$$\frac{AB}{PQ} = \frac{AL}{PM}$$

$$\frac{AB^2}{PQ^2} = \frac{AL^2}{PM^2} \quad \text{... (2)}$$

From (1) and (2), $\dfrac{\text{Area of } \Delta ABC}{\text{Area of } \Delta PQR} = \dfrac{AL^2}{PM^2}$

Result (2) : The ratio of the areas of two similar triangles is equal to the ratio of the squares of their corresponding angle bisectors.

Given : Let Δ ABC and Δ PQR be the given triangles such that $\Delta ABC ||| \Delta PQR$ and AD, PE are bisectors of $\angle A$ and $\angle P$ respectively.



To prove : $\dfrac{\text{Area of } \Delta ABC}{\text{Area of } \Delta PQR} = \dfrac{AD^2}{PE^2}$

Proof : We know that, the ratio of the areas of two similar triangles is equal to the ratio of squares of any pair of their corresponding sides.

$$\therefore \frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta PQR} = \frac{AB^2}{PQ^2} \qquad \text{... (1)}$$

By the given data,

$\Delta ABC \,|||\, \Delta PQR$

$\Rightarrow \quad \angle A = \angle P$

$\Rightarrow \quad \frac{1}{2} \angle A = \frac{1}{2} \angle P$

$\Rightarrow \quad \angle BAD = \angle QPE$

In ΔABD and ΔPQE

$\angle BAD = \angle QPE$ and $\angle B = \angle Q$

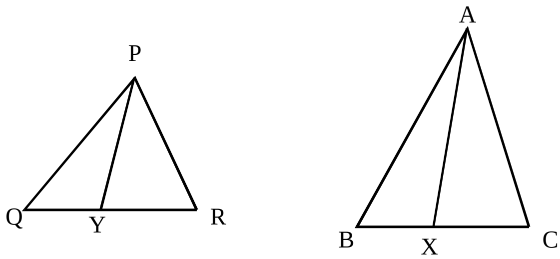
By AA similarity, $\Delta ABD \,|||\, \Delta PQE$

$$\therefore \frac{AB}{PQ} = \frac{AD}{PE} \Rightarrow \frac{AB^2}{PQ^2} = \frac{AD^2}{PE^2}$$

From (1) and (2), $\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta PQR} = \frac{AD^2}{PE^2}$

Result (3) : The ratio of the areas of two similar triangles is equal to the ratio of the squares of their corresponding medians.

Given : Let Δ ABC and ΔPQR be the given triangles such that $\Delta ABC \,|||\, \Delta PQR$ and AX, PY are medians of Δ ABC, Δ PQR respectively.



To prove : $\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta PQR} = \frac{AX^2}{PY^2}$

Proof : We know that, the ratio of the areas of two similar triangles is equal to the ratio of squares of any pair of their corresponding sides.

$$\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta PQR} = \frac{AB^2}{PQ^2} \qquad \text{... (1)}$$

By the given data, Δ ABC ||| Δ PQR $\Rightarrow \frac{AB}{PQ} = \frac{BC}{QR} = \frac{2BX}{2QY} = \frac{BX}{QY}$

In Δ ABP and Δ DEQ, $\angle B = \angle Q$ [since Δ ABC ||| Δ PQR]

and $\frac{AB}{PQ} = \frac{BX}{QY}$

By SAS - similarity, Δ ABX ||| Δ PQY

$$\therefore \quad \frac{AB}{PQ} = \frac{AX}{PY} \Rightarrow \frac{AB^2}{PQ^2} = \frac{AX^2}{PY^2} \qquad \text{... (2)}$$

From (1) and (2), $\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta PQR} = \frac{AX^2}{PY^2}$

7.11 Theorem

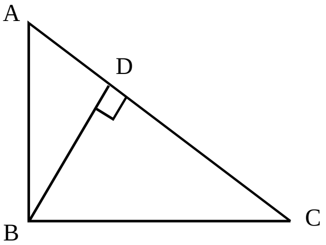
If a perpendicular is drawn from the vertex of the right angle of a right triangle to the hypotenuse, the triangles on each side of the perpendicular are similar to the whole (original) triangle and to each other. [Proof is not required]

(i.e.,) If ABC is a triangle right angled at B and BD $\perp$ AC, then

(i) Δ ADB ||| Δ ABC

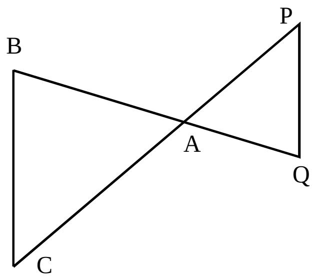
(ii) Δ BDC ||| Δ ABC

(iii) Δ ADB ||| Δ BDC



Example 7.26 : In fig, Δ ABC ||| Δ AQP. If BC = 8 cm, PQ = 4 cm, BA = 6.5 cm, AP = 2.8 find CA and AQ.

Solution : By the given data, Δ ABC ||| Δ AQP



$$\Rightarrow \quad \frac{AB}{AQ} = \frac{BC}{QP} = \frac{AC}{AP}$$

$$\Rightarrow \quad \frac{AC}{AP} = \frac{BC}{QP} \quad \text{and} \quad \frac{BC}{QP} = \frac{AB}{AQ}$$

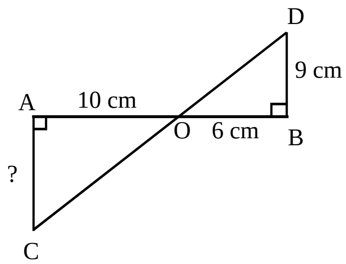
$$\Rightarrow \quad \frac{AC}{2.8} = \frac{8}{4} \quad \text{and} \quad \frac{8}{4} = \frac{6.5}{AQ}$$

$\Rightarrow$ AC = 5.6 cm and AQ = 3.25 cm.

Example 7.27 : In fig, CA and DB are perpendicular to AB. If AO = 10 cm, BO = 6 cm and DB = 9 cm, find AC.

Solution : In Δ OAC and Δ OBD

$\angle OAC = \angle OBD = 90^o$



$\angle AOC = \angle BOD$ (vertically opposite angles)

By AA criterion of similarity

Δ AOC ||| Δ BOD

$$\therefore \frac{AO}{BO} = \frac{OC}{OD} = \frac{AC}{BD} \Rightarrow \frac{AO}{BO} = \frac{AC}{BD}$$

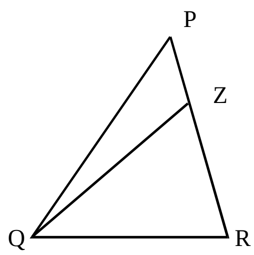
$$\Rightarrow \frac{10}{6} = \frac{AC}{9} \Rightarrow AC = 15 \text{ cm.}$$

Example 7.28 : PQR is a triangle in which PQ = PR and Z is a point on the side PR such that QR^2 = PR.RZ. Prove that QZ = QR.

Solution : Given : Δ PQR in which PQ = PR, Z is a point on PR such that QR^2 = PR.RZ.

To prove : QZ = QR

Proof : Given : QR^2 = PR.RZ



$$\Rightarrow \quad \frac{QR}{RZ} = \frac{PR}{QR}$$

In Δ PQR and Δ ZQR,

$\angle R$ is common and $\frac{QR}{RZ} = \frac{PR}{QR}$

Δ PQR ||| Δ QZR $\quad \therefore \frac{PQ}{QZ} = \frac{QR}{ZR} = \frac{PR}{QR}$

$$\Rightarrow \quad \frac{PQ}{QZ} = \frac{PR}{QR} \Rightarrow QZ = QR \quad \text{(since PQ = PR)}$$

Example 7.29 : The perimeter of two similar triangles ABC and PQR are respectively 36 cm and 48 cm. If PQ = 16 cm find AB.

Solution : We know that the ratio of the corresponding sides of similar triangles is same as the ratio of their perimeters.

Here Δ ABC ||| PQR.

$$\Rightarrow \qquad \frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR} = \frac{36}{48}$$

$$\Rightarrow \qquad \frac{AB}{PQ} = \frac{36}{48} \qquad \Rightarrow \qquad \frac{AB}{16} = \frac{36}{48}$$

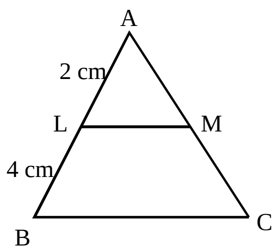
$$\Rightarrow \qquad AB = \frac{3}{4} \times 16 \qquad \Rightarrow \qquad AB = 12 \text{ cm.}$$

Example 7.30 : L and M are points on sides AB and AC of a Δ ABC. If AL = 2 cm LB = 4 cm and LM || BC prove that 3 LM = BC.

Solution : In Δ ALM and Δ ABC

Since LM || BC, ∠ALM = ∠ABC and ∠AML = ∠ACB
(corresponding angles of || lines)

∠A is common



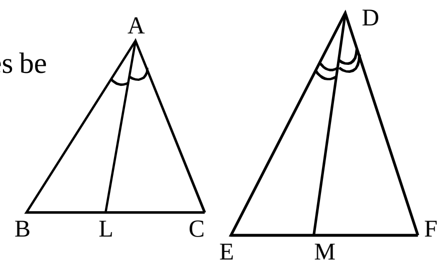
By AAA criterion, Δ ALM ||| Δ ABC

$$\therefore \quad \frac{AL}{AB} = \frac{LM}{BC} = \frac{AM}{AC}$$

$$\Rightarrow \qquad \frac{AL}{AB} = \frac{LM}{BC} \Rightarrow \frac{2}{6} = \frac{LM}{BC} \Rightarrow BC = 3\,LM.$$

Example 7.31 : If two triangles are equiangular. Prove that the ratio of the corresponding sides is the same as the ratio of the corresponding angle bisector segments.

Solution : Let the two given triangles be ABC and DEF.



By the given data

∠A = ∠D, ∠B = ∠E, ∠C = ∠F

Let AL and DM be the bisectors of A and D respectively.

Now to prove : $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF} = \frac{AL}{DM}$

By AAA - similarity, Δ ABC ||| Δ DEF

$$\therefore \quad \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} \quad \ldots (1)$$

$$\Rightarrow \text{Now } \angle A = \angle D \Rightarrow \frac{\angle A}{2} = \frac{\angle D}{2} \Rightarrow \angle BAL = \angle EDM$$

In Δ BAL and Δ EDM

$\angle B = \angle E$ (given)

$\angle BAL = \angle EDM$ (proved)

By AA similarity, Δ ABL ||| Δ DEM

$$\therefore \frac{AB}{DE} = \frac{BL}{EM} = \frac{AL}{DM}$$

$$\Rightarrow \quad \frac{AB}{DE} = \frac{AL}{DM} \quad \ldots (2)$$

From (1) and (2), $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = \frac{AL}{DM}$

Example 7.32 : Δ ABC and Δ DEF are similar, the area of Δ ABC is 16 sq.cm and that Δ DEF is 25 sq.cm. If EF = 4 cm find BC.

Solution : By the given condition, Δ ABC ||| Δ DEF

We know that the ratio of areas of two similar triangles is equal to ratio of squares of their corresponding sides,

$$\therefore \frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta PQR} = \frac{BC^2}{EF^2}$$

$$\Rightarrow \quad \frac{16}{25} = \frac{BC^2}{4^2} \Rightarrow BC^2 = \frac{16 \times 16}{25}$$

$$BC = \frac{16}{5} = 3.2 \text{ cm.}$$

Example 7.33 : Prove that the area of the equilateral triangle described on the side of a square is half the area of the equilateral triangle described on its diagonal.

Solution : Let PQRS be the given square.

Equilateral triangle STR has been described on the side SR and equilateral triangle PXR has been described on the diagonal PR.

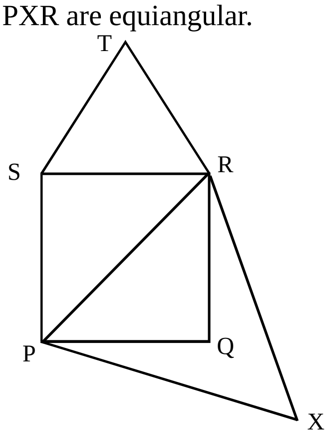
In right angled Δ PSR

$$PR^2 = PS^2 + SR^2$$

$$= SR^2 + SR^2 \qquad \text{(since PS = SR)}$$

$$PR^2 = 2SR^2 \ldots (1)$$

Now equilateral Δ STR and equilateral Δ PXR are equiangular.



$\Rightarrow \quad \Delta$ SRT ||| Δ PRX

$$\Rightarrow \quad \frac{\text{Area of } \Delta SRT}{\text{Area of } \Delta PRX} = \frac{SR^2}{PR^2}$$

$$= \frac{SR^2}{2SR^2} \quad \text{[from (1)]}$$

$$\Rightarrow \text{Area of } \Delta SRT = \frac{1}{2} \text{ [area of } \Delta PRX]$$

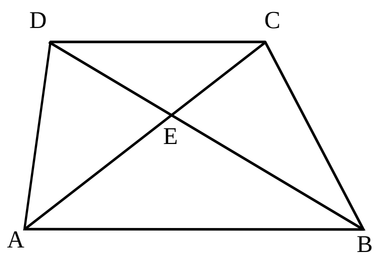
Example 7.34 : ABCD is a trapezium with AB || DC. The diagonal AC and BD intersect at E. If Δ AED ||| Δ BEC. Prove that AD = BC.

Solution : By the given data ABCD is a trapezium and AB || DC.

In Δ ECD and Δ ABE, $\angle EDC = \angle EBA$

$\angle ECD = \angle EAB$

By AA - similarity, Δ DEC ||| Δ BEA



$$\therefore \frac{DE}{BE} = \frac{EC}{EA} = \frac{DC}{BA} \Rightarrow \frac{DE}{BE} = \frac{EC}{EA}$$

$$\Rightarrow \quad \frac{DE}{EC} = \frac{BE}{EA} \quad \ldots (1)$$

Also given Δ DEA ||| Δ CEB

$$\therefore \frac{DE}{CE} = \frac{EA}{EB} = \frac{DA}{CB} \quad ... (2)$$

From (1) and (2), $\frac{BE}{EA} = \frac{EA}{EB}$

$\Rightarrow \quad EB^2 = EA^2 \Rightarrow EB = EA$

Substituting EB = EA in (2) we get

$$\frac{EA}{EA} = \frac{DA}{CB} \Rightarrow DA = CB \Rightarrow AD = BC$$

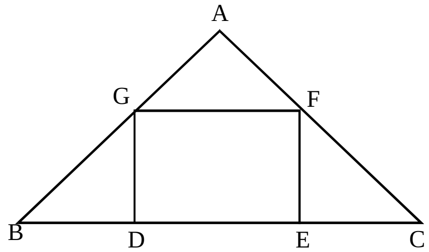
Example 7.35 : In fig. DEFG is a square and $\angle BAC = 90^o$. Prove that (i) Δ AGF ||| Δ DBG, (ii) Δ AGF ||| Δ EFC, (iii) Δ DBG ||| Δ EFC.

Solution : (i) In Δ AGF and Δ DBG

$\angle GAF = \angle BDG = 90^o$

$\angle AGF = \angle DBG$ (corresponding angles of parallel lines)

By AA - similarity, ΔAGF ||| ΔDBG



(ii) In Δ AGF and Δ EFC

$\angle FAG = \angle CEF = 90^o$

and $\angle AFG = \angle ECF$

By AA - similarity, Δ AGF ||| Δ EFC

From (i) and (ii)

Δ AFG ||| Δ DBG and Δ AGF ||| Δ EFC

Δ DBG ||| Δ EFC.

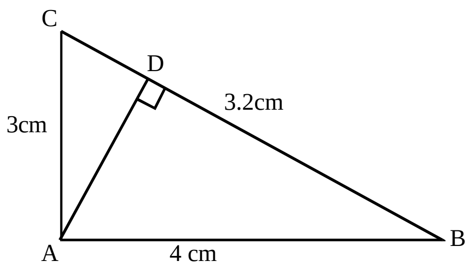
Example 7.36 : In fig. ABC is a triangle, right angled at A and AD $\perp$ BC. If AC = 3 cm, AB = 4 cm, BD = 3.2 cm, find AD.

Solution : In Δ ABC, $\angle A = 90^o$

and AD $\perp$ BC

By theorem 7.11

Δ CAB ||| Δ ADB



$$\Rightarrow \quad \frac{CA}{AD} = \frac{AB}{DB} = \frac{BC}{BA}$$

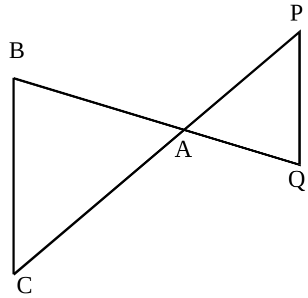
$$\Rightarrow \quad \frac{CA}{AD} = \frac{AB}{DB}$$

$$\Rightarrow \quad \frac{3}{AD} = \frac{4}{3.2}$$

$$\Rightarrow \quad AD = \frac{3 \times 3.2}{4}$$

$$\Rightarrow \quad AD = 2.4 \text{ cm.}$$

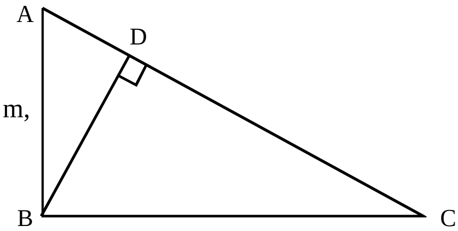
EXERCISE 7.5



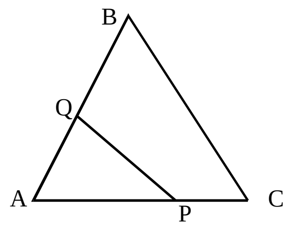
1. In fig. Δ ABC ||| Δ AQP (i) if BC = 8 cm, PQ = 4 cm, BA = 6.5 cm and AP = 2.8 cm, find CA and AQ. (ii) If BC = 4 cm, PQ = 3 cm, AP = 5.7 cm, AQ = 3.6 cm, find AB and AC.
2. A vertical stick 10 cm long casts a shadow 8 cm long. At the same time a tower casts the shadow 30 m long. Determine the height of the tower.
3. In Δ ABC, X is a point on AB and Y is a point on AC such that XY || BC. Find the length of XY if AX = 1 cm, XB = 3 cm and BC = 6 cm.
4. Δ ABC is a right angled triangle, right angled at B and BD ⊥ AC. If BD = 8 cm, AD = 4 cm find CD.
5. In fig. ∠ABC = 90°, and BD ⊥ AC.

   (i) If AB = 75 cm, BC = 1 m, CD = 1.25 m, find BD.

   (ii) If AB = 5.7 cm, BD = 3.8 cm, CD = 5.4 cm, find BC.



6. In the adjoining figure two triangles APQ and ABC are similar. If PQ and BC are not parallel and PC = 4 cm, AQ = 3 cm, QB = 12 cm and BC = 15 cm and AP = PQ calculate (i) the length of AP, (ii) the ratio of the area of Δ APQ and Δ ABC.



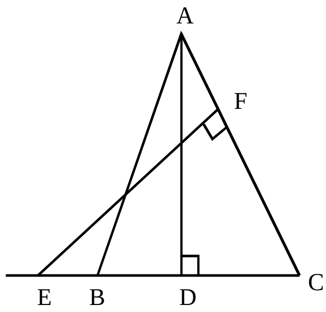
7. If two sides and a median bisecting one of these sides of a triangle are respectively proportional to the two sides and the corresponding median of another triangle then prove that the triangles are similar.
8. Two poles of height a metres and b metres are p metres apart. Prove that the height of the point of intersection of the line joining the top of each pole to the foot of the opposite pole is given by $\frac{ab}{a+b}$ metres.

9. Prove that the ratio of the corresponding altitudes of two similar triangles is equal to the ratio of their corresponding sides.

10. Δ ABC and Δ DEF are similar, the area of Δ ABC is 9 sq.cm. and that of Δ DEF is 16 sq.cm. If EF = 4.2 cm, find BC.

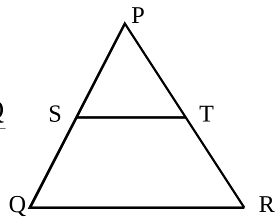
11. The areas of two similar triangles are 16 cm^2 and 36 cm^2 respectively. If the altitude of the first triangle is 3 cm, find the corresponding altitude of the other.

12. In the figure, E is a point on side CB produced of an isosceles triangle ABC with AB = AC. If AD $\perp$ BC and EF $\perp$ AC prove that Δ ABD ||| Δ ECF.



13. In the figure, ST || QR and $\frac{PS}{SQ} = \frac{3}{5}$ calculate the value of

(i) $\frac{\text{Area of } \Delta \text{PST}}{\text{Area of } \Delta \text{ PQR}}$ (ii) $\frac{\text{Area of trapezium STRQ}}{\text{Area of } \Delta \text{ PQR}}$



14. The perimeter of two similar triangles ABC and PQR are respectively 36 cm and 24 cm. If PQ = 10 cm find AB.

15. One angle of a triangle is equal to one angle of another and bisectors of these equal angles divide the opposite sides in the same ratio, prove that the triangles are similar.

16. D, E, F are the mid points of the sides BC, CA and AB respectively of triangle ABC. Determine the ratio of the areas of triangles ABC and DEF.

17. Prove that if the diagonals of a quadrilateral cut each other in the same ratio, the quadrilateral is a trapezium.

18. Two circles touch at O straight lines AOP, BOS, COT cut one circle at A, B, C and cut the other circle at P, S, T. Prove that area of Δ ABC : Area of Δ PST = $AB^2 : PS^2$.

19. In ΔABC, PQ is a line segment intersecting AB at P and AC at Q such that PQ || BC and PQ divides ΔABC into two equal parts in area. Find $\frac{BP}{AB}$.

8. CO-ORDINATE GEOMETRY

8.1 INTRODUCTION :

Rene Descartes (1596 - 1650) is credited with the invention of co-ordinate geometry, since he used a rectangular coordinate system to establish a relationship between equations and curves. Descartes' best-known work is a philosophical treatise published in 1637, called **A Discourse on the method of Rightly conducting the Reason and Seeking Truth in the Science.**

This publication had three appendices, the third of which is the famous La geometrie. In this appendix Descartes deals with such topics as the solution of equations of degree greater than 2 and the case of exponents to indicate powers of numbers.

But what Descartes is best remembered for is his introduction of modern analytical geometry in **La geometrie** and the Cartesian co-ordinate system is named in his honour.

Although Descartes is credicted with the development of analytical geometry, Pierre de Fermat, another great French Mathematician also formulated co-ordinate geometry at the same period and made considerable contribution in this field.

The modern terms coordinates, abscissa and ordinate were contributed by the German Mathematician, Gottfried Wilhelm Von Leibniz in 1692.

We have learnt the following. Let us recall them.

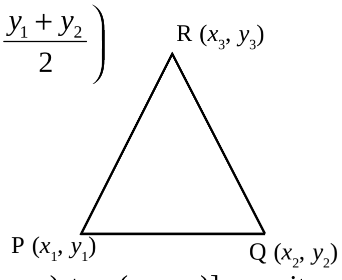
(i) Distance between two points A (x_1, y_1) and B (x_2, y_2) ;

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \text{ units.}$$

(ii) Mid-point formula : If M is the mid-point of the line segment joining

A (x_1, y_1) and B (x_2, y_2) then M is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

8.2 AREA OF A TRIANGLE



Let P (x_1, y_1), Q (x_2, y_2) and R (x_3, y_3) be the vertices of the Δ PQR.

The area of the triangle = $\frac{1}{2}[x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2)]$ sq.units

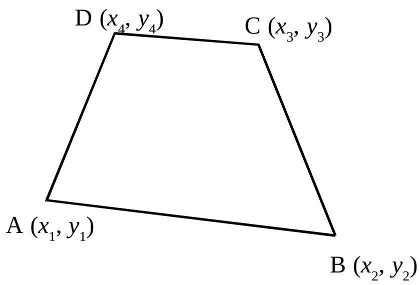
There is a short-cut method to find the area of the above triangle.

Step 1 : Arrange the vertices as follows :

$$\begin{matrix} x_1 & & x_2 & & x_3 & & x_1 \\ & \times & & \times & & \times & \\ y_1 & & y_2 & & y_3 & & y_1 \end{matrix}$$

Step 2 : Area = $\frac{1}{2}$ $[(x_1 y_2 + x_2 y_3 + x_3 y_1) - (x_2 y_1 + x_3 y_2 + x_1 y_3)]$

We have to take the vertices in the anticlockwise direction. [Otherwise we will get a negative value. Let us take it as positive]



One advantage in using this shortcut method is that we need not divide the quadrilateral into two triangles, while finding its area.

Area of the quadrilateral ABCD

$$= \frac{1}{2}\left[\begin{matrix} x_1 & & x_2 & & x_3 & & x_4 & & x_1 \\ & \times & & \times & & \times & & \times & \\ y_1 & & y_2 & & y_3 & & y_4 & & y_1 \end{matrix}\right]$$

$$= \frac{1}{2}[(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1) - (x_2y_1 + x_3y_2 + x_4y_3 + x_1y_4)]$$

Of course, we have taken the vertices in the **anticlockwise direction** and that too **in order.** This method can be extended to any polygon.

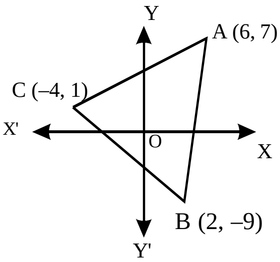
If they are not given in order, we have to arrange them in proper order.

It is better, we plot them roughly on a diagram as shown in examples.

Example 8.1 : Find the area of the triangle ABC given A (6, 7), B (2, –9) and C (–4, 1).

Solution : Let us take the order as Δ ACB.

$$\text{Area of the } \Delta \text{ ABC} = \frac{1}{2}\left[\begin{matrix} 6 & & -4 & & 2 & & 6 \\ & \times & & \times & & \times & \\ 7 & & 1 & & -9 & & 7 \end{matrix}\right]$$

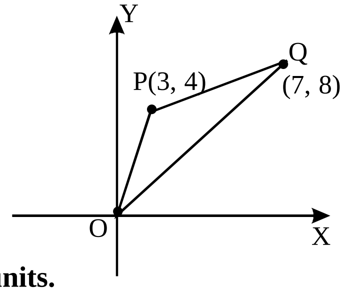


$$= \frac{1}{2}\ [(6 + 36 + 14) - (-28 + 2 - 54)]$$

$$= \frac{1}{2}\ [56 + 80] = \frac{1}{2} \times 136 = \textbf{68 sq.units.}$$

Note : If one of the vertices is the origin, then the formula becomes $\frac{1}{2}\,[x_1y_2 - x_2y_1]$.

Example 8.2 : Find the area of the Δ POQ given P (3, 4), Q (7, 8). O is the origin.



Solution : Area $= \frac{1}{2}\,[x_1y_2 - x_2y_1]$

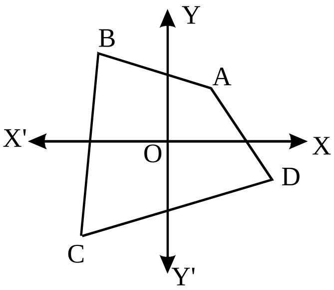
$= \frac{1}{2}\,[(7 \times 4) - (3 \times 8)]$

$= \frac{1}{2}\,[28 - 24] =$ **2 sq.units.**

Example 8.3 : Find the area of the quadrilateral ABCD given A (1, 2), B (–3, 4), C (–5, –6) and D (4, –1)

Solution : We can see that the points are taken in order anticlockwise.

$\therefore$ Area of the quadrilateral ABCD



$$= \frac{1}{2}\begin{bmatrix} 1 & -3 & -5 & 4 & 1 \\ 2 & 4 & -6 & -1 & 2 \end{bmatrix}$$

$= \frac{1}{2}\,[(4 + 18 + 5 + 8) \; - (-6 - 20 - 24 - 1)]$

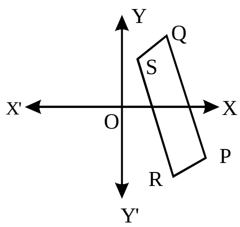
$= \frac{1}{2}\,[35 + 51] = \frac{1}{2} \times 86 =$ **43 sq.units.**

Example 8.4 : Find the area of the quadrilateral PQRS given P (5, –2), Q (3, 4), R (4, –7), S (1, 1).

Solution : Let us plot them roughly since the points are not in order.

Let us arrange them in anticlockwise order (i.e.,) P(5,–2), Q(3,4), S(1,1), R(4,–7)

Area of the quadrilateral PQRS $= \frac{1}{2}\begin{bmatrix} 5 & 3 & 1 & 4 & 5 \\ -2 & 4 & 1 & -7 & -2 \end{bmatrix}$



$= \frac{1}{2}\,[(20 + 3 - 7 - 8) - (-6 + 4 + 4 - 35)]$

$= \frac{1}{2}\,[8 + 33] = \frac{41}{2} =$ **20.5 sq.units.**

Condition for collinearity of three points

Let us consider the three points A (x_1, y_1), B (x_2, y_2) and C (x_3, y_3).

We know that the area of the Δ ABC is

$$\Delta = \frac{1}{2}\ (x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)]$$

If A, B and C lie on a line, then the area of the Δ ABC becomes zero.

Hence, the condition for the three points, to be collinear is $(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3) = 0$ (or) $(x_1y_2 + x_2y_3 + x_3y_1) = (x_2y_1 + x_3y_2 + x_1y_3)$

Example 8.5 : Prove that the points (4, 5), (6, –1) and (0, 17) are collinear.

Solution : $x_1 = 4, y_1 = 5, x_2 = 6, y_2 = -1, x_3 = 0, y_3 = 17$

LHS : $x_1y_2 + x_2y_3 + x_3y_1$

$= (4 \times -1) + (6 \times 17) + (0 \times 5) = -4 + 102 = 98$

RHS : $x_2y_1 + x_3y_2 + x_1y_3$

$= (6 \times 5) + (0 \times -1) + (4 \times 17) = 30 + 68 = 98.$

We see LHS = RHS.

Hence, they are collinear.

Example 8.6 : Determine the value of '*a*' such that (a, a), (2, 3), (4, –1) are collinear.

Solution : $x_1 = a, x_2 = 2, x_3 = 4 ; y_1 = a, y_2 = 3, y_3 = -1$

Condition for collinearity

$$\begin{aligned} x_1y_2 + x_2y_3 + x_3y_1 &= x_2y_1 + x_3y_2 + x_1y_3 \\ 3a - 2 + 4a &= 2a + 12 - a \\ 7a - a &= 12 + 2 \\ 6a &= 14 \Rightarrow a = \frac{7}{3} \end{aligned}$$

EXERCISE 8.1

1. Find the area of each of the following triangular whose vertices are given :

(i) (8, 10), (8, 4), (–4, 4)
(ii) (–2, 4), (0, –3), (6, –1)
(iii) (4, 5), (1, 3), (0, –4)
(iv) origin, (8, 0), (0, 10)
(v) origin, (4, 8), (6, 4)

2. Find the area of each of the following quadrilaterals whose vertices are :

(i) (6, –1), (5, –6), (2, 2), (4, 5) (ii) (3, 4), (–1, 6), (–3, –4), (6, 1)
(iii) (6, 9), (7, 4), (4, 2), (3, 7) (iv) (–2, –3), (0, 4), (–4, 2), (0, –1)

3. Show that each of the following points are collinear.

(i) (4, –1), (–2, 17), (2, 5) (ii) (2, 6), (–2, 2), (–1, 3)
(iii) (12, 5), (6, 2), (2, 0) (iv) (–2, 1), (1, 2) (4, 3)

4. The vertices of some Δs are given along with their areas. Find '*a*' in each of the following.

Vertices	Area
(i) (2, 3), (6, –2), (–2, *a*)	6
(ii) (3, 8), (4, *a*), (5, –2)	8
(iii) (–2, –1), (2, 1), (*a*, –3)	5
(iv) (*a*, –3), (3, 1), (–1, –5)	2

5. Find the value of '*m*' in each of the following, given that they are collinear.

(i) (–3, 2), (–5, 6), (*m*, –2) (ii) (5, 7), (9, *m*), (3, 6)
(iii) (1, 4), (7, *m*), (3, 5) (iv) (2, –3), (0, 1), (*m*, *m*)

6. Show that each of the following points are collinear. (**Hint :** Take two pairs of 3 points. Prove that they are collinear)

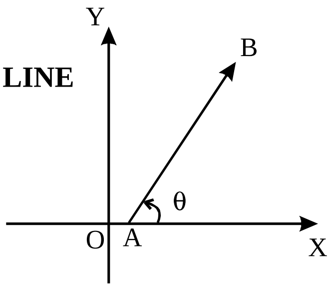
(i) (4, 8), (–4, 0), (–3, 1), (–7, –3), (ii) (2, 5), (–2, 1), (3, 6), (0, 3)

7. If the points (2, 5), (4, 6) and (8, *a*) lie on a straight line, find the value of '*a*'.

8. If the points (–1, 3), (*b*, –1), (0, 4) are on a line, find the value of '*b*'.

9. If the point (*x*, *y*) is collinear with the points (*a*, 0) and (0, *b*), then prove that $\frac{x}{a} + \frac{y}{b} = 1$.

8.3 SLOPE OR GRADIENT OF A STRAIGHT LINE



Let a straight line cut the X axis at A. The angle θ between the straight line and the positive direction of the X axis when measured in the anti-clockwise direction is called the angle of inclination. The tangent of the angle of inclination is called the **slope or gradient** of the line.

In the figure AB is the portion of the line above the X axis and if ∠XAB = θ then slope = tanθ. The slope of the line is denoted by *m*.

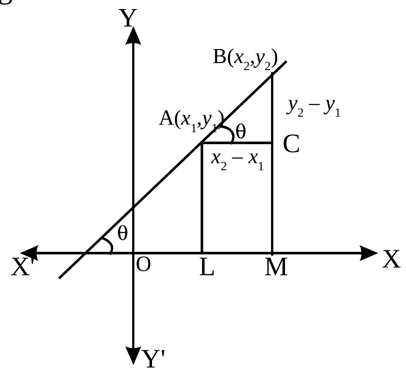
Note : 1. The slope of the line parallel to the X axis is zero, Since $\tan 0^o = 0$.

2. If a line is perpendicular to the X axis, its inclination is 90^o and so its slope is $\tan 90^o$ which is not defined.

8.3.1 Slope of a line joining two points

Let A (x_1, y_1) and B (x_2, y_2) be the two given points. Let θ be the inclination of the line AB with positive direction of the X axis.

Draw horizontal and vertical lines parallel to X axis and Y axis respectively through A and B intersecting at C. In right angled Δ ABC.

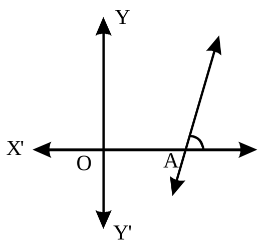


$$\tan\theta = \frac{BC}{AC} = \frac{BM - CM}{OM - OL}$$

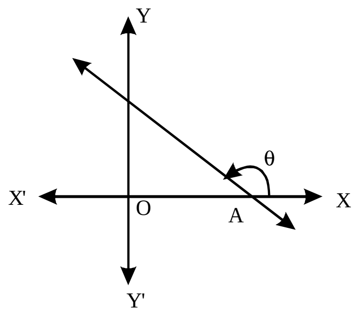
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\therefore \text{ slope } = \frac{y_2 - y_1}{x_2 - x_1} \text{ or } \frac{y_1 - y_2}{x_1 - x_2}$$

The following figures show different positions of lines and their slopes.

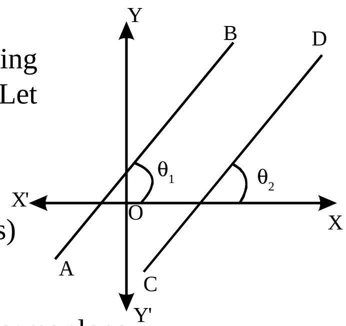


Slope = tan θ
since θ is acute,
slope is positive



Slope = tan θ
since θ is obtuse,
slope is negative

8.3.2 Condition for two lines to be parallel



Let AB and CD be any two parallel lines making angles θ_1 and θ_2 with the positive direction of X axis. Let their slopes be m_1 and m_2 respectively.

$\therefore$ $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$

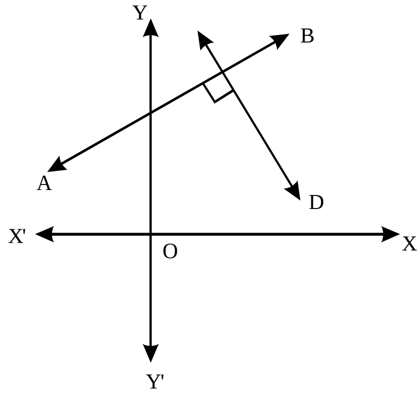
Since AB || CD, $\theta_1 = \theta_2$ (corresponding angles)

$\therefore$ $\tan\theta_1 = \tan\theta_2$ (i.e.,) $m_1 = m_2$

Hence, if two lines are parallel, they have the same slope.

8.3.3 Condition for two lines to be perpendicular

If the product of the slopes of two lines is –1, then the lines are perpendicular. If m_1 and m_2 are the slopes of two lines which are perpendicular then $\boldsymbol{m_1 m_2 = -1}$.



Note :

1. If the line is parallel to X axis (perpendicular to Y axis), then the slope is 0.
2. If the line is perpendicular to X axis (parallel to Y axis), then the slope is not defined.
3. If the slope of a line is 'm', then the slope of the line perpendicular to it is $-\frac{1}{m}$.

Example 8.7 : Find the slope of the line, given that it makes (i) 0^o, (ii) 45^o (iii) 60^o with X-axis.

Solution : (i) $\theta = 0^o$

Slope $= \tan\theta = \tan 0^o = 0$

(ii) $\theta = 45^o$

Slope $= \tan\theta = \tan 45^o = 1$

(iii) $\theta = 60^o$

Slope $= \tan\theta = \tan 60^o = \sqrt{3}$

Example 8.8 : Find the angle of inclination of the line whose slope is (i) 1 (ii) $\sqrt{3}$.

Solution : (i) Let θ be the angle of inclination of the line

$$\text{slope} = 1$$

$$\tan\theta = 1$$

$$\tan\theta = \tan 45$$

$$\theta = 45$$

$$\text{(ii) Slope} = \sqrt{3}$$

$$\tan\theta = \sqrt{3}$$

$$\tan\theta = \tan 60^o$$

$$\theta = 60^o$$

Example 8.9 : Find the slope of the line joining the points (1, 3) and (4, 6)

Solution : Let A (1, 3) and B (4, 6) be the given points

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{6-3}{4-1} = \frac{3}{3} = 1$$

Example 8.10 : Find the slope of a line which is (i) parallel to a line whose slope is $\frac{1}{2}$ (ii) perpendicular to a line whose slope is –3.

Solution : (i) Slope of the given line = $\frac{1}{2}$

$\therefore$ Slope of its parallel line is $\frac{1}{2}$

(ii) m = Slope of the given line = –3

Slope of its perpendicular line is $\frac{-1}{m} = \frac{1}{3}$

Example 8.11 : Find the slope of a line which is perpendicular to a line passing through (0, 6) and (–3, 1)

Solution : Let P (0, 6) and Q (–3, 1) be the given points.

$$\text{Slope of PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-6}{-3-0} = \frac{-5}{-3} = \frac{5}{3}$$

Slope of the line perpendicular to PQ = $\frac{-3}{5}$ [⊥r line's slope]

Example 8.12 : Find a if the slope of the line joining (–6, 13) and (3, a) is $-\frac{1}{3}$

Solution : $\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$

$$\text{Slope} = \frac{a-13}{3+6}$$

$$\Rightarrow \quad \frac{-1}{3} = \frac{a-13}{9}$$

$$\Rightarrow \quad -9 = 3a - 39$$

$$\Rightarrow \quad a = 10$$

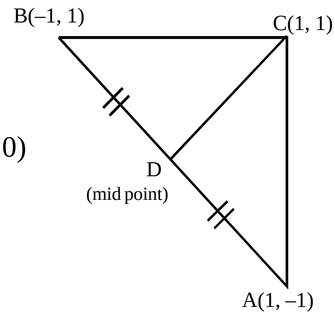
Example 8.13 : A triangle has vertices at (1, –1), (–1, 1) and (1, 1). Find the slope of the median through C.

Solution : Let A (1, –1), B (–1, 1) and C (1, 1) be the given points.

$$\text{Mid point} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$\text{D is the mid point of AB} = \left(\frac{1-1}{2}, \frac{-1+1}{2}\right) = (0, 0)$$

$$\text{Slope of CD} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-1}{0-1}$$



[Here C(1, 1); D(0, 0),
$x_1 = 1, y_1 = 1$; $x_2 = 0, y_2 = 0$]

Slope of its median CD = 1

Example 8.14 : A straight line passes thorugh (3, –4) and (–4, 3). Another line has slope 1. Are the lines parallel or perpendicular ?

Solution : Let the given points be P (3, –4) and Q (–4, 3).

$$m_1 = \text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Slope of PQ } (m_1) = \frac{3+4}{-4-3} = \frac{7}{-7} = -1$$

Given that slope of another line (m_2) = 1

$$m_1 \times m_2 = -1 \times 1$$

$$= -1$$

Since $m_1 \times m_2 = -1$, the given lines are perpendicular.

Example 8.15 : The line joining A (–2, 4) and B (3, –5) is parallel to the line joining C (0, 4) and D (–3, *y*) find '*y*'.

Solution : $\text{Slope of AB} = \frac{-5-4}{3+2} = \frac{-9}{5}$

$$\text{Slope of CD} = \frac{y-4}{-3-0} = \frac{y-4}{-3}$$

Now Slope of AB = Slope of CD [Since AB || CD]

$$\Rightarrow \quad \frac{-9}{5} = \frac{y-4}{-3}$$

$$\Rightarrow \quad 27 = 5y - 20$$

$$\Rightarrow \quad 27 + 20 = 5y$$

$$\Rightarrow \quad y = \frac{47}{5}$$

Example 8.16 : The line joining A (–4, 6) and B (–1, –3) is perpendicular to the line joining C (0, –4) and D (3, *a*), find *a*.

Solution : $m_1 = \text{Slope of AB} = \frac{-3-6}{-1+4} = \frac{-9}{3}$

$$m_2 = \text{Slope of CD} = \frac{a+4}{3-0} = \frac{a+4}{3}$$

$$m_1 \times m_2 = -1 \qquad \text{[Since the lines are perpendicular]}$$

$$-3\left(\frac{a+4}{3}\right) = -1$$

$$-a-4 = -1$$

$$-a = -1 + 4 = 3$$

$$\therefore a = -3$$

Example 8.17 : Without using distance formula, show that the points P (3, 2), Q (0, –1), R (–3, –2) and S (0, 1) are the vertices of a parallelogram.

Solution : P (3, 2), Q (0, –1), R (–3, –2) and S (0, 1) are the given points

$$\text{Slope of PQ} = \frac{-1-2}{0-3} = \frac{-3}{-3} = 1$$

$$\text{Slope of QR} = \frac{-2+1}{-3-0} = \frac{-1}{-3} = \frac{1}{3}$$

$$\text{Slope of RS} = \frac{1+2}{0+3} = \frac{3}{3} = 1$$

$$\text{Slope of SP} = \frac{2-1}{3-0} = \frac{1}{3}$$

Slope of PQ = Slope of RS = 1

$\therefore$ PQ || RS.

Also, Slope of QR = Slope of SP = $\frac{1}{3}$ (Verify)

$\therefore$ QR || SP.

Since the opposite sides are parallel, the given points form a parallelogram.

Example 8.18 : A (–1, 2), B (2, 1) and C (0, 4) are the vertices of a triangle ABC. Find the slope of altitude of AB.

Solution : A (–1, 2), B (2, 1) and C(0, 4) are the vertices of a triangle ABC

$$m = \text{Slope of AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-2}{2+1} = \frac{-1}{3}$$

$$\therefore\ m = \frac{-1}{3}$$

$$\text{Slope of altitude of AB} = -\frac{1}{m} = -\left(\frac{1}{-\frac{1}{3}}\right) = 3$$

Example 8.19 : A (2, 3), B (0, 4), C (–5, 0) are the given points. Find the slope of the line passing through B and mid point of AC.

Solution : Let the mid point of AC be D.

$$\text{Mid point of AC} = \left(\frac{2-5}{2}, \frac{3+0}{2}\right)$$

$$\text{D is } \left(\frac{-3}{2}, \frac{3}{2}\right)$$

$$\text{Slope of BD} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\frac{3}{2} - 4}{-\frac{3}{2} - 0} = \frac{-\frac{5}{2}}{-\frac{3}{2}} = \frac{5}{3}$$

Example 8.20 : Show that the points (–2, 3), (3, 4) and (8, 5) are collinear.

Solution : Let A (–2, 3), B (3, 4) and C (8, 5) be the given points.

$$\text{Slope of AB} = \frac{4-3}{3+2} = \frac{1}{5}$$

$$\text{Slope of BC} = \frac{5-4}{8-3} = \frac{1}{5}$$

Slope of AB = Slope of BC.

$\therefore$ AB || BC. Since B is the common point, AB coincides with BC.
Hence A, B, C are collinear.

Example 8.21 : If the points (4, 1), (–2, –3) and (x, –5) are collinear, find x.

Solution : Let A (4, 1), B (–2, –3) and C (x, –5) be the given points.

$$\text{Slope of AB} = \frac{-3-1}{-2-4} = \frac{-4}{-6} = \frac{2}{3}$$

$$\text{Slope of BC} = \frac{-5+3}{x+2} = \frac{-2}{x+2}$$

Since the points are collinear, slope are equal.

$$\text{Slope of AB} = \text{Slope of BC}$$

$$\frac{2}{3} = \frac{-2}{x+2}$$

$$2x + 4 = -6$$

$$\therefore x = -5$$

Example 8.22 : The vertices of a Δ ABC are A (–3, 3), B (–1, –4) and C (5, –2). M and N are the mid points of AB and AC. Show that MN is parallel to BC and MN = $\frac{1}{2}$ BC.

Solution : Mid point of AB $= \left(\frac{-3-1}{2}, \frac{3-4}{2}\right)$, $\therefore$ M is $\left(-2, \frac{-1}{2}\right)$

Mid point of AC $= \left(\frac{-3+5}{2}, \frac{3-2}{2}\right)$, $\therefore$ N is $\left(1, \frac{1}{2}\right)$

$$\text{Slope of MN} = \frac{\frac{1}{2}+\frac{1}{2}}{1+2} = \frac{1}{3}$$

$$\text{Slope of BC} = \frac{-2+4}{5+1} = \frac{2}{6} = \frac{1}{3}$$

Since the slopes are equal, the line segments MN and BC are parallel.

$\therefore$ MN || BC

$$\text{MN} = \sqrt{(1+2)^2+\left(\frac{1}{2}+\frac{1}{2}\right)^2} = \sqrt{(3)^2+(1)^2} = \sqrt{9+1} = \sqrt{10}$$

$$\text{BC} = \sqrt{(5+1)^2+(-2+4)^2} = \sqrt{(6)^2+(2)^2} = \sqrt{40} = 2\sqrt{10}$$

$$\text{Thus MN} = \frac{1}{2}\text{BC}.$$

EXERCISE - 8.2

1. Find the slope of the line whose inclination with X axis is

   (i) 30^o, (ii) 45^o, (iii) 60^o, (iv) 90^o (v) α

2. Find the slope of the line joining the points

   (i) (4, 5) with the origin, (ii) (1,4) and (2, –3) , (iii) $\left(\frac{2}{3}, \frac{5}{3}\right)$ and $\left(\frac{5}{3}, \frac{3}{2}\right)$,

   (iv) (3, $\sqrt{3}$) with the origin. (v) ($\sin\theta$, $-\cos\theta$) and ($-\sin\theta$, $\cos\theta$)

3. OB and OC are two straight lines through the origin O, making angles 30^o and 60^o with OX. What is the slope of the bisector of $\angle$BOC.

4. Find the slopes of PQ and RT with the points P(–1, 2), Q (–8, 2), R (4, –1) and T(–3, 5).

5. A (2, 3), B (0, 4), C (–5, 0) are three given points. Find the slope of the line through B and the mid point of AC.

6. Show that the points A, B and C are collinear when the coordinates are

   (i) A (4, 1), B (–2, –3), C (–5, –5)

   (ii) A (–2, 5), B (0, –1), C (1, –4)

7. If the three points (2, *a*), (4, 6) and (8, 8) are collinear find *a*.

8. If the three points (3, –1), (*x*, 3) and (1, –3) are collinear find *x*.

9. A triangle has vertices at (3, 4), (1, 2) and (–5, –6) find the slope of its median.

10. Show that the following points taken in order form the vertices of a parallelogram.

    (i) (–2, –1), (1, 0), (4, 3), (1, 2), (ii) (1, –2), (–1, 4), (5, 8), (7, 2).

11. Find the slope of the line parallel to the line joining the points (3, 5) and (5, 8).

12. Find *p* if the slope of a line joining (–5, 15) and (4, *p*) is –1/9.

13. The line joining A (–2, 3) and B (–1, 5) is parallel to C (0, 5) and D (–2, *y*), find *y*.

14. The line joining A (–1, –2) and B (5, 6) is perpendicular to C (4, 2) and D (0, *y*) find *y*.

15. P, Q, R have co-ordinates (–2, 1), (2, 2) and (6, –2) find (i) Gradient of the line parallel to PQ. (ii) Gradient of the line perpendicular to QR.

16. Find the slope of the perpendicular bisector of the line AB where A and B are the points (–2, –3), (4, 5) respectively.

17. What is the slope of the line perpendicular to the line joining A (5, 1) and M where M is the mid point of the segment joining (4, 2) and (–6, 4).

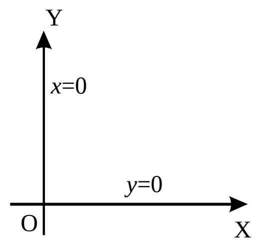
18. The vertices of a triangle are A (1, 8), B (–2, 4) and C (8, –5) M and N are the mid points of AB and AC. Show that MN is parallel to BC and MN = $\frac{1}{2}$ BC.

19. The vertices of a triangle ABC are A (3, 2), B (4, 5) and C (–2, –4). Find the slope of the line EF where E and F are the mid points of AB and AC respectively. Prove that EF is parallel to BC.

20. The vertices of a triangle ABC are A (1, 2), B (–4, 5) and C (0, 1). Find the slope of the altitudes.

8.4 EQUATION OF A LINE

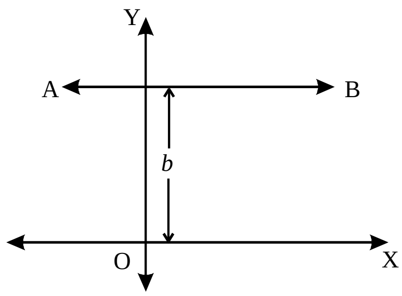
8.4.1 Equations of coordinate axes



The x coordinate of every point on OY is 0. Therefore equation of OY is $x = 0$.

The y coordinate of every point on OX is 0. Therefore the equation of OX is $y = 0$.

Equation of a straight line parallel to X axis.

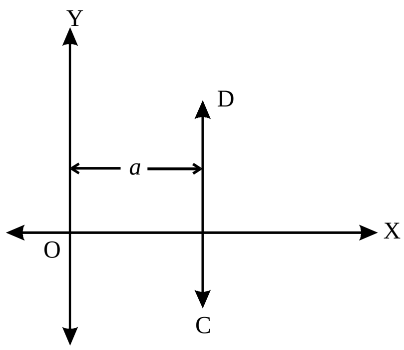


Let AB be a straight line parallel to the X axis and at distance 'b' from X-axis. Then y coordinate of every point on AB is b.

The equation of AB is $y = b$

Equation of the line parallel to X axis is $y = b$.

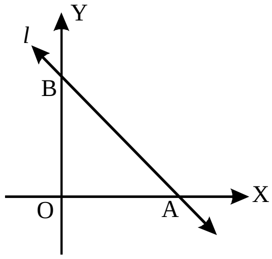
Equation of a straight line parallel to the Y axis.



Let CD be a straight line parallel to Y axis and at a distance 'a' from Y axis. Then X coordinate of every point on CD is 'a'.

The equation of CD is $x = a$. Equation of the line parallel to Y axis is $x = a$.

Intercepts



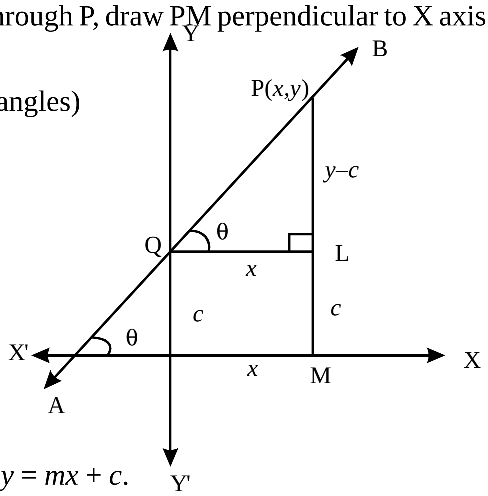
If a straight line l meets the X axis at A and Y axis at B, then OA, the distance of A from the origin is known as **X-intercept** and OB, the distance of B from the origin is known as **Y-intercept.**

8.4.2 Slope-intercept form

To find the equation of a straight line when slope 'm' and Y intercept 'c' are given.

Let AB be a straight line which makes an angle θ with OX and cuts OY at Q. Let OQ = c. Slope of the line $\tan\theta = m$

Let P (x, y) be any point on AB, through P, draw PM perpendicular to X axis and QL perpendicular to MP.



Then ∠PQL = θ (corresponding angles)

From right angled Δ PQL

$$\tan\theta = \frac{PL}{QL}$$

$$\text{Slope} = \frac{y - c}{x}$$

$$\Rightarrow \quad mx = y - c$$

$$mx + c = y$$

∴ Required equation of a line is $y = mx + c$.

8.4.3 Slope-point form

To find the equation of a line passing through a point (x_1, y_1) and with given slope 'm'.

The equation of the line having slope 'm' is $y = mx + c$. ... (1)

where c is an unknown constant. Since the line (1) passes through (x_1 , y_1), the coordinates (x_1, y_1) will satisfy the equation.

$$\therefore y_1 = mx_1 + c \qquad ... (2)$$

Subtracting (2) from (1)

$$y - y_1 = mx + c - mx_1 - c$$

$$y - y_1 = mx - mx_1$$

$$y - y_1 = m(x - x_1)$$

Required equation of a line is $y - y_1 = m(x - x_1)$

Result : Equation of the **straight line passing through the origin** and having slope m is $y = mx$.

8.4.4 Two-points form

To find the equation of a straight line passing through two points (x_1, y_1) and (x_2, y_2).

Let the straight line passes through two fixed points A (x_1, y_1) and B (x_2, y_2).

Slope of the line AB is $\dfrac{y_2 - y_1}{x_2 - x_1} = m$

$\therefore$ Slope of the line is m, it passes through A (x_1, y_1)

Equation of line is $y - y_1 = m(x - x_1)$

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

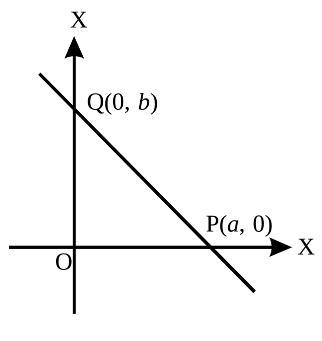
$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$\therefore$ Equation of the straight line passing through (x_1, y_1) and (x_2, y_2) is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

8.4.5 Intercepts form

Equation of a straight line which makes intercepts *a* and *b* on the coordinate axes is $\dfrac{x}{a} + \dfrac{y}{b} = 1$



Let PQ be the given line which intercepts the X axis at P $(a, 0)$ and the Y axis at Q $(0, b)$. OP is called the X intercept and OQ is called the Y intercept of the line PQ.

$\therefore$ Required equation is the equation to PQ.

Hence, equation of PQ is $\frac{y-0}{b-0} = \frac{x-a}{0-a}$

$$\Rightarrow \quad \frac{y}{b} = \frac{x-a}{-a}$$

$$\Rightarrow \quad \frac{y}{b} = -\frac{x}{a} + 1$$

$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 1$ is the equation of straight line in intercept form.

Example 8.23 : Find the equation of a straight line whose (i) slope is 4 and y intercept is –3. (ii) Inclination is 30° and y intercept is 5.

Solution : Slope (m) = 4

y intercept (c) = –3

Equation of a line is $y = mx + c$

$y = 4x - 3$

Equation of a line is $4x - y - 3 = 0$

(ii) $\theta = 30^{o}$, y intercept = 5

slope = $\tan\theta$

$m = \tan 30^{o} = \frac{1}{\sqrt{3}}$

Equation of a line is $y = mx + c$

$$y = \frac{1}{\sqrt{3}} x + 5$$

$$\sqrt{3}\, y = x + 5\sqrt{3}$$

Equation of a line is $x - \sqrt{3}\, y + 5\sqrt{3} = 0$.

Example 8.24 : Calculate the slope and y intercept of the line $2x - 3y + 1 = 0$.

Solution : $2x - 3y + 1 = 0$

$3y = 2x + 1$

$$y = \frac{2x}{3} + \frac{1}{3}$$

Comparing with $y = mx + c$, we get

$$m = \frac{2}{3}, c = \frac{1}{3}$$

$$\text{slope} = \frac{2}{3}; \quad y \text{ intercept} = \frac{1}{3}$$

Example 8.25 : Does the straight line $4x - 3y - 8 = 0$ pass through the point (2, 0)

Solution : If the line $4x - 3y - 8 = 0$ passes through the point (2, 0) then $x = 2$, $y = 0$ must satisfy its equation.

When $x = 2$, $y = 0$

$$4x - 3y - 8 = 4(2) - 3(0) - 8 = 8 - 8 = 0$$

$\therefore$ (2, 0) satisfies the given equation.

$\therefore$ The line pass through the point (2, 0).

Example 8.26 : A straight line passes through (1, 2) and has the equation $y = 2x + k$ find k.

Solution : If the line passes through (1, 2) then $x = 1$ and $y = 2$ must satisfy the equation $y = 2x + k$.

$$2 = 2(1) + k$$

$$\therefore k = 0$$

Example 8.27 : Find the equation of a straight line passing through (–4, 5) and having slope $-\frac{2}{3}$.

Solution : Slope $= -\frac{2}{3}$

Point is (–4, 5)

Equation of the line is $(y - y_1) = m(x - x_1)$

$$y - 5 = -\frac{2}{3}(x + 4)$$

$$3y - 15 = -2x - 8$$

$\therefore$ Equation of a line is $2x + 3y - 7 = 0$

Example 8.28 : If a line passes through the mid point of AB where A is (3, 0) and B is (5, 4) and makes an angle 60° with X axis find its equation.

Solution : Mid point $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$

$$\text{Mid point of AB} = \left(\frac{3+5}{2}, \frac{0+4}{2} \right) = (4, 2)$$

$$\theta = 60^o$$

$$\text{Slope} = m = \tan 60^o = \sqrt{3}$$

Equation of a line is $(y - y_1) = m\,(x - x_1)$

$$y - 2 = \sqrt{3}\,(x - 4)$$

$$y - 2 = \sqrt{3}\,x - 4\sqrt{3}$$

Equation of a line is $\sqrt{3}\,x - y + 2 - 4\sqrt{3} = 0$.

Example 8.29 : Find the equation of the straight line passing through the points (3, 6) and (–2, 5).

Solution : Equation of the line is $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$

$$\frac{y-6}{5-6} = \frac{x-3}{-2-3}$$

$$\frac{y-6}{-1} = \frac{x-3}{-5}$$

$$5y - 30 = x - 3$$

$$x - 5y - 3 + 30 = 0$$

Equation of a line is $x - 5y + 27 = 0$

Example 8.30 : A (–5, 2), B (4, –6) and C (1, 7) are the vertices of Δ ABC. Find the equation of the median drawn from A to BC.

Solution : Let D be the mid point of BC.

$$\text{Mid point of BC} = \left(\frac{4+1}{2}, \frac{-6+7}{2} \right)$$

$$D \text{ is } \left(\frac{5}{2}, \frac{1}{2}\right)$$

Equation of the median AD is $\dfrac{y-y_1}{y_2-y_1} = \dfrac{x-x_1}{x_2-x_1}$

$$\frac{y-2}{\frac{1}{2}-2} = \frac{x+5}{\frac{5}{2}+5}$$

$$\Rightarrow \quad \frac{y-2}{-\frac{3}{2}} = \frac{x+5}{\frac{15}{2}}$$

$$\Rightarrow \quad \frac{y-2}{-1} = \frac{x+5}{5}$$

$$\Rightarrow \quad 5y - 10 = -x - 5$$

$$\Rightarrow \quad x + 5y - 10 + 5 = 0$$

$$\Rightarrow \quad x + 5y - 5 = 0$$

Equation of the median is $x + 5y - 5 = 0$

Example 8.31 : Find the equation of the straight line joining the points (7, 5) and (9, 7). Hence show that the points (4, 2) (7, 5) and (9, 7) are collinear.

Solution : Let A (7, 5), B (9, 7) and C (4, 2) be the given points.

Equation of the line is $\dfrac{y-y_1}{y_2-y_1} = \dfrac{x-x_1}{x_2-x_1}$

Equation of the line AB is $\dfrac{y-5}{7-5} = \dfrac{x-7}{9-7}$

$$\Rightarrow \quad \frac{y-5}{2} = \frac{x-7}{2}$$

$$\Rightarrow \quad y - 5 = x - 7$$

$$\Rightarrow \quad x - y - 7 + 5 = 0$$

Equation of the line is $x - y - 2 = 0$

Substituting $x = 4$ and $y = 2$ in this equation $x - y - 2 = 0$

Now $x - y - 2 = 4 - 2 - 2 = 0$

$\therefore$ The equation is satisfied.

Hence (4, 2) lies on the line joining (7, 5) and (9, 7).

$\therefore$ The given points are collinear.

Example 8.32: Find the equation of the line cutting off intercepts $\frac{-4}{3}$ and $\frac{3}{4}$ on the X and Y axes respectively.

Solution : X intercept (a) = $\frac{-4}{3}$

Y intercept (b) = $\frac{3}{4}$

Equation of the line is $\frac{x}{a} + \frac{y}{b} = 1$

$$\frac{x}{\frac{-4}{3}} + \frac{y}{\frac{3}{4}} = 1$$

$$\frac{-3x}{4} + \frac{4y}{3} = 1$$

$$-9x + 16y = 12$$

$$9x - 16y = -12$$

Equation of the line is $9x - 16y + 12 = 0$

Example 8.33 : Find the equation of the line passing through (5, –3) and whose intercepts on the axes are equal in magnitude but opposite in sign.

Solution : Let the intercepts be a and $-a$.

Equation of the line is $\frac{x}{a} + \frac{y}{b} = 1$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{-a} = 1$$

$$\Rightarrow \quad x - y = a$$

The line passes through (5, –3)

$$\therefore\ 5 + 3 = a$$

$$a = 8$$

Required equation is $x - y - 8 = 0$

Example 8.34 : Find the intercepts cut off by the line $2x - 3y + 5 = 0$ on the axes.

Solution : x intercept : put $y = 0$

$$\therefore\ 2x + 5 = 0$$

$$x = \frac{-5}{2} \quad \text{This is the } x\text{-intercept}$$

y intercept : put $x = 0$

$$-3y + 5 = 0$$

$$\therefore\ y = \frac{5}{3} \quad \text{This is the } y\text{-intercept}$$

Example 8.35 : The intercepts made by the line $4x - 5y = 20$ on the X axis is 'a'. Find a.

Solution : Equation of the line is $4x - 5y = 20$

This line intersects the X axis, when y coordinate is zero.

$$\therefore\ 4x - 0 = 20$$

$$x = 5$$

$$x\text{-intercept} = 5$$

But x-intercept is a

$$\therefore\ a = 5$$

Example 8.36 : Find the equation of the line which passes through the point (3, 4) and makes intercepts on the axes such that their sum is 14.

Solution : Equation of the line is $\frac{x}{a} + \frac{y}{b} = 1$

$$\text{Sum of the intercepts} = 14$$

$$a + b = 14$$

$$b = (14 - a)$$

Equation of the line is $\frac{x}{a} + \frac{y}{14-a} = 1$

$$(14 - a)\,x + ay = a\,(14 - a) \quad \ldots (1)$$

This line passes through (3, 4)

$$(14 - a)\,3 + 4a = a\,(14 - a)$$

$$\Rightarrow \quad a^2 - 13a + 42 = 0$$

$$\Rightarrow \quad (a - 7)(a - 6) = 0$$

$$a = 7 \text{ or } a = 6$$

Substituting $a = 7$ in (1)

$$(14 - 7)\,x + 7y = 7\,(14 - 7)$$

$$\Rightarrow \quad 7x + 7y = 49$$

$$\Rightarrow \quad x + y = 7$$

Sub. $a = 6$ in (1)

$$(14 - 6)\,x + 6y = 6\,(14 - 6)$$

$$\Rightarrow \quad 8x + 6y = 48$$

$$\Rightarrow \quad 4x + 3y = 24$$

Equation of the line is $x + y - 7 = 0$ or $4x + 3y - 24 = 0$.

EXERCISE - 8.3

1. Find the equation of the straight line whose (i) slope is $\frac{1}{3}$ and y intercept is 2.

   (ii) Slope is 4 and y intercept is $\frac{5}{2}$

   (iii) inclination is 60^o and y intercept is 2.

2. Find the slope and the y intercept of the following lines (i) $2x + 3y - 5 = 0$, (ii) $3x = 2y$.

3. Find the equation of the line having an inclination 45^o with the positive direction of X axis and cut off an intercept of 4 on the positive side of Y axis.

4. Prove that the point (1, 0) always lies on the line $x + ky - 1 = 0$ for all values of k.

5. Find the equation of a line whose slope is $-\frac{1}{2}$ and passing through (1, –2).

6. A line passes through the mid point of the join of (2, –3) and (–2, 1)) and is inclined to the X axis at the angle θ where tanθ = 3/4 find the equation of the line.

7. Find the equation of the line passing through the points (–2, –5) and (0, 6).

8. A (2, –4), B (3, 3) and C (–1, 5) are the vertices of triangle ABC. Find the equation of the median of the triangle through A.

9. Find the equation to the line passing through the points (1, 4) and (3, –2). Hence show that points (1, 4) and (3, –2) and (–3, 16) are collinear.

10. Find the X and Y intercept of the line $2x - 3y - 12 = 0$.

11. A line $x + 2y = 2$ cuts the coordinate axes at A and B, find the length of AB.

12. Find the equation in intercept form of the line joining (4, –3) and (–2, 6).

13. Find the equation of the line passing through (3, –4) and making equal intercepts on the axes.

14. Find the equation of the line passing through (–3, 10) and making intercepts a, b on the X, Y axis whose sum is 8.

15. A line intersects X axis at (–2, 0) and cuts off an intercept of 3 from the positive side of Y axis write the equation of the line.

16. A straight line AB cuts the coordinate axes X and Y at A and B. If the mid point of AB is (2, 3) find the equation of AB.

8.5 GENERAL FORM OF A STRAIGHT LINE

The linear equation $ax + by + c = 0$ always represents a straight line. This is the general form of a straight line.

Slope of the line $ax + by + c = 0$:

$$ax + by + c = 0$$

$$by = -ax - c$$

$$y = \frac{-a}{b}x - \frac{c}{b}$$

Comparing this with the form $y = mx + c$

$$\text{Slope of the line} = \frac{-a}{b}$$

$$\text{In general, Slope} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } y}$$

8.5.1 Condition for parallelism of two straight lines

We know that two straight lines are parallel, if their slopes are equal. If m_1 and m_2 are the slopes of two parallel lines then $m_1 = m_2$.

The equation of all lines parallel to the line $ax + by + c = 0$ can be put in the form $ax + by + k = 0$ for different values of k.

8.5.2 Condition for perpendicularity of two lines

We know that two lines are perpendicular if the product of their slopes is –1. If m_1 and m_2 are the slopes of two perpendicular lines, then $m_1 \times m_2 = -1$.

The equation of all lines perpendicular to the line $ax + by + c = 0$ can be written as $bx - ay + k = 0$ for different values of k.

Example 8.37 : Find the slope of the line $4x = 3y - 2$

Solution : $4x - 3y + 2 = 0$; Slope $= -\dfrac{\text{Coefficient of } x}{\text{Coefficient of } y} = -\left(\dfrac{4}{-3}\right) = \dfrac{4}{3}$

Example 8.38 : Show that the line joining (–2, 2) and (–5, –6) and the line joining (4, –8) and (7, 0) are parallel.

Solution : Slope of the line joining the points (–2, 2) and (–5, –6)

$$m_1 = \frac{-6-2}{-5+2} = \frac{-8}{-3} = \frac{8}{3}$$

Slope of the line joining the points (4, –8) and (7, 0)

$$m_2 = \frac{0+8}{7-4} = \frac{8}{3}$$

Since $m_1 = m_2$, the two lines are parallel.

Example 8.39 : Are the lines $5x + 3y + 3 = 0$ and $3x - 5y + 7 = 0$ perpendicular?

Solution : Slope $= -\dfrac{\text{Coefficient of } x}{\text{Coefficient of } y}$

Slope of the line $5x + 3y + 3 = 0 = (m_1) = \dfrac{-5}{3}$

Slope of the line $3x - 5y + 7 = 0 = (m_2) = -\left(\dfrac{3}{-5}\right) = \dfrac{3}{5}$

$$\text{Thus } m_1 \times m_2 = \frac{-5}{3} \times \frac{3}{5} = -1$$

Since $m_1 \times m_2 = -1$, the given lines are perpendicular.

Example 8.40 : If m is the slope of a line parallel to $3x + 4y + 7 = 0$ find the value of $16m^2$.

Solution : $$\text{Slope} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } y} = -\frac{3}{4}$$

$\therefore$ Slope of its parallel line, $m = \dfrac{-3}{4}$

$$16m^2 = 16\left(\frac{-3}{4}\right)^2 = 16\left(\frac{9}{16}\right) = 9$$

$$16m^2 = 9$$

Example 8.41 : Find the equation of a line parallel to X axis and passing through the point (–4, 7).

Solution : Equation of a line parallel to X axis is $y = b$ ... (1)

Since it passes through (–4, 7)

$$\therefore 7 = b \quad \text{... (2)}$$

from (1) and (2), we get $y = 7$

$\therefore$ Equation of the required line is $y = 7$.

Example 8.42 : If $3x - 2y + 4 = 0$ and $ax + 4y + 3 = 0$ are parallel lines, find the value of 'a'.

Solution : Slope of the line $3x - 2y + 4 = 0 = (m_1) = -\left(\dfrac{3}{-2}\right) = \dfrac{3}{2}$

Slope of the line $ax + 4y + 3 = 0$ $(m_2) = \dfrac{-a}{4}$

Since the two lines are parallel, $m_1 = m_2$.

$$\frac{3}{2} = \frac{-a}{4}$$

$$\therefore a = -6$$

Example 8.43 : Find the equation of the line through the point P (–3, 1) and parallel to the line joining the points Q (6, –1) and R (–2, –5).

Solution : Slope of QR $= \dfrac{-5+1}{-2-6} = \dfrac{-4}{-8} = \dfrac{1}{2}$

Slope of line parallel to QR $= \dfrac{1}{2}$

This line passes through P (–3, 1)

Equation of a line is $(y - y_1) = m\,(x - x_1)$

$$\Rightarrow \quad y - 1 = \frac{1}{2}(x + 3)$$

$$\Rightarrow \quad 2y - 2 = x + 3$$

$$\Rightarrow \quad x - 2y + 5 = 0$$

Equation of a line is $x - 2y + 5 = 0$

Example 8.44 : Find the equation of the line passing through (–2, 3) and perpendicular to the line joining the points (6, 7) and (0, –1).

Solution : Slope of the line joining (6, 7) and (0, –1) $= \dfrac{-1-7}{0-6} = \dfrac{4}{3}$

$\therefore$ Perpendicular line's slope $= \dfrac{-3}{4}$

$\therefore$ Required equation passes through (–2, 3) whose slope is $\dfrac{-3}{4}$

$$\therefore \quad \text{Equation is } y - 3 = \frac{-3}{4}(x + 2)$$

$$\Rightarrow \quad 3x + 4y - 6 = 0$$

Example 8.45 : A (4, 1), B (7, 4) and C (5, –2) are the vertices of Δ ABC, find the equation of the altitude through A.

Solution : Slope of BC $= \dfrac{-2-4}{5-7} = \dfrac{-6}{-2} = 3$

$\therefore$ Slope of the altitude through A $= -\dfrac{1}{3}$

It passes through (4, 1)

Equation of the altitude is $y - y_1 = m(x - x_1)$

$$y - 1 = -\frac{1}{3}(x - 4)$$

$$\Rightarrow \quad 3y - 3 = -x + 4$$

$$\Rightarrow \quad x + 3y - 7 = 0$$

Equation of the altitude is $x + 3y - 7 = 0$.

Example 8.46 : Find the equation of the perpendicular bisector of the line joining the points A (–1, 2) and B (3, –4).

Solution : Let D be the mid point of AB.

$$\text{Mid point of AB} = \left(\frac{-1+3}{2}, \frac{2-4}{2}\right)$$

$$\text{D is } (1, -1)$$

$$\text{Slope of AB} = \frac{-4-2}{3+1} = \frac{-6}{4} = \frac{-3}{2}$$

Slope of perpendicular bisector = $\frac{2}{3}$

It passes through D (1, –1)

Equation of line is $y - y_1 = m(x - x_1)$

$$\Rightarrow \quad y + 1 = \frac{2}{3}(x - 1)$$

$\Rightarrow 2x - 3y - 5 = 0$ which is the equation of the perpendicular bisector.

Example 8.47 : Write down the equation of the line parallel to $x - 2y + 8 = 0$ and passing through the point (1, 2).

Solution : Slope of the line $x - 2y + 8 = 0$ is $\frac{-1}{-2} = \frac{1}{2}$

$\therefore$ Slope of its parallel line is $\frac{1}{2}$

This line passes through (1, 2)

Equation of required line is $(y - y_1) = m(x - x_1)$

$$\Rightarrow \quad y - 2 = \frac{1}{2}(x - 1)$$

$$\Rightarrow \quad 2y - 4 = x - 1$$

$$\Rightarrow \quad x - 2y + 3 = 0$$

***Aliter* :** Any line parallel to $x - 2y + 8 = 0$ is of the form $x - 2y + k = 0$

This line passes through (1, 2)

$$1 - 2(2) + k = 0$$

$$\therefore k = 3$$

Equation of required line is $x - 2y + 3 = 0$.

Example 8.48 : Is the line through (–2, 3) and (4, 1) perpendicular to the line $3x - y - 1 = 0$? Does the line $3x - y - 1 = 0$ bisect the join of (–2, 3) and (4, 1).

Solution : Let A (–2, 3) and B (4, 1) be the given points

$$\text{Slope of AB} = \frac{1-3}{4+2} = \frac{-2}{6} = m_1 \text{ (say)}$$

$$m_1 = \frac{-1}{3}$$

$$\text{Slope of the given line} = \frac{-3}{-1}$$

$$m_2 = 3$$

$$m_1 \times m_2 = \frac{-1}{3} \times 3 = -1$$

Since $m_1 \times m_2 = -1$, the lines are perpendicular.

$$\text{Mid point of AB} = \left(\frac{-2+4}{2}, \frac{3+1}{2}\right) = \left(\frac{2}{2}, \frac{4}{2}\right) = (1, 2)$$

Substituting (1, 2) in the equation $3x - y - 1 = 0$

$$3x - y - 1 = 3(1) - 2 - 1 = 0$$

(i.e.,) The equation is satisfied.

$\therefore$ The given line bisects the join of two points.

Example 8.49 : The foot of perpendicular from (1, 2) on a line is the origin, find the equation of the line.

Solution : Let A (1, 2) and B (0, 0) be the given points

$$\text{Slope of AB} = \frac{0-2}{0-1} = 2$$

Slope of its perpendicular line is $-\frac{1}{2}$

This line passes through (0, 0)

Equation of a line is $y - y_1 = m(x - x_1)$

$$\Rightarrow \quad y - 0 = -\frac{1}{2}(x - 0)$$

$$\Rightarrow \quad x + 2y = 0 \quad \text{which is the required equation.}$$

Example 8.50 : Find the equation of the straight line perpendicular to the line $4x - 3y + 2 = 0$ and which passes through (–2, 3).

Solution : Equation of the straight line which is perpendicular to $4x - 3y + 2 = 0$ is of the form $3x + 4y + k = 0$.

This passes through (–2, 3)

$$\therefore -6 + 12 + k = 0$$

$$\therefore k = -6$$

$\therefore$ The required equation of the line is $3x + 4y - 6 = 0$.

EXERCISE - 8.4

1. Show that the straight lines $2x - 5y + 1 = 0$ and $6x - 15y = 4$ are parallel.
2. Show that the straight lines $7x - y + 6 = 0$ and $3x + 21y + 11 = 0$ are perpendicular to each other.
3. If $4x - 5y = 3$ and $bx - 4y = 2$ are perpendicular to each other, find the value of b.
4. Lines $2x - \mathrm{b}y + 5 = 0$ and $ax + 3y = 2$ are parallel. Find the relation connecting a and b.
5. Write down the equation of a line parallel to $3x - 4y - 5 = 0$ and passing through the point (2, 3).

6. A line passes through the point (–3, –2) and (5, –1). Find the equation of the straight line parallel to it and passing through (2, 5).

7. Write down the equation of the line perpendicular to $3x + 8y = 12$ and passing through the point (–1, –2).

8. If the coordinates of A, B, C are (1, –2) (3, 1) and (–2, 3) respectively. Find the equation of the perpendicular from A to BC.

9. The vertices A, B, C of triangle are (2, 1), (6, –1), (4, 11) find the equation of the altitude through A.

10. Find the equation of the perpendicular bisector of the line joining the points (i) (–2, 3) and (2, –1) (ii) (1, 3) and (–2, 2), (iii) (0, 0) and (6, 4).

11. Find the equation of the line at a distance

    (i) 4 units from X axis and parallel to it.

    (ii) –3 units from Y axis and parallel to it.

12. Find the equation of the line given that it passes through the point (–3, –4) and parallel of X axis.

13. Show that the line $x + y - 2 = 0$ is perpendicular bisector of the line joining the origin and (2, 2).

14. Find the equation of the line which cuts the Y axis at a distance of 6 units from origin and perpendicular to $2x + y = 8$.

15. The foot of the perpendicular from the point (1, 1) to a line is the point (–3, 4). Find the equation of the line.

16. Write down the equation of the line AB through (3, 2) perpendicular to the line $3x - 2y + 5 = 0$. AB meets the X axis at A and the Y axis at B. Calculate the area of triangle OAB where O is the origin.

8.6 INTERSECTION OF TWO STRAIGHT LINES

We know that the two lines if not parallel in a plane intersect in a unique point. Thus if the equation of two lines are given we can find the point of intersection by solving these equations simultaneously for x and y and then their point of intersection is the ordered pair (x, y).

Example 8.51 : Obtain the coordinates of the point of intersection of the lines $2x - y + 5 = 0$ and $x + y + 1 = 0$.

Solution : $2x - y = -5$ ... (1)

$x + y = -1$ ... (2)

(1) + (2) ⇒ $3x = -6$

$\therefore \; x = -2$

Substitute $x = -2$ in (2)

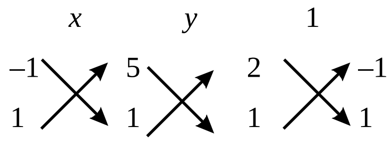
$$-2 + y = -1$$

$$y = 1$$

Point of intersection is (–2, 1).

ANOTHER METHOD :

We use rule of cross-multiplication to solve equations. The working rule is given below.



$$\frac{x}{-1-5} = \frac{y}{5-2} = \frac{1}{2+1}$$

$$x = \frac{-6}{3} = -2 \; ; \; y = \frac{3}{3} = 1$$

$\therefore$ Point of intersection is (–2, 1).

Example 8.52 : Find the equation of the line passing through (3, 4) and the point of intersection of the lines $x + 2y = 3$ and $2x - y = 1$.

Solution : $x + 2y = 3$ ... (1)

$2x - y = 11$ ... (2)

(1) $\Rightarrow$ $x + 2y = 3$

(2) × 2 $\Rightarrow$ $4x - 2y = 2$

Adding $5x = 5$

$x = 1$

Substitute $x = 1$ in (1) we get $1 + 2y = 3$

$\therefore y = 1$

Point of intersection is (1, 1)

Given point is (3, 4). Required equation is the line joining (1, 1) and (3, 4)

$$\frac{y-1}{4-1} = \frac{x-1}{3-1}$$

$$\Rightarrow \quad \frac{y-1}{3} = \frac{x-1}{2}$$

$$\Rightarrow \quad 2y - 2 = 3x - 3$$

$$\therefore \quad 3x - 2y - 1 = 0 \text{ is the required line.}$$

Example 8.53 : Find the length of the line segment which joins the point of intersection of the lines $x - 2y + 3 = 0$, $2x - 3y + 4 = 0$ and the mid point of the line joining the points (3, 6) and (5, 4).

Solution : Point of intersection of the two lines $x - 2y + 3 = 0$ and

$2x - 3y + 4 = 0$ is (1, 2) [Verify yourself]

Now Mid point of (3, 6) and (5, 4) = $\left(\frac{3+5}{2}, \frac{6+4}{2}\right)$ = (4, 5)

Length of the line segment joining (1, 2) and (4, 5) = $\sqrt{(4-1)^2 + (5-2)^2}$

$$= \sqrt{(3)^2 + (3)^2} = \sqrt{18} \text{ units.}$$

Example 8.54 : Where does the line $3x + 2y = 8$ meet (i) the X axis, (ii) Y axis.

Solution : $3x + 2y = 8$

$$\Rightarrow \quad \frac{3x}{8} + \frac{2y}{8} = \frac{8}{8}$$

$$\Rightarrow \quad \frac{x}{8/3} + \frac{y}{4} = 1$$

Thus its intercepts are $\frac{8}{3}$ and 4.

$\therefore$ The given line meets X axis at $\left(\frac{8}{3}, 0\right)$ and Y axis at (0, 4).

Example 8.55 : A line joining the points (2, 3) and (–1, 2) meets the line $3x - 4y + 11 = 0$ at A. Find the co-ordinates of A.

Solution : Given points are (2, 3) and (–1, 2)

Equation of a line joining (2, 3) and (–1, 2) is $\dfrac{y-y_1}{y_2-y_1} = \dfrac{x-x_1}{x_2-x_1}$

$$\frac{y-3}{2-3} = \frac{x-2}{-1-2}$$

$$\frac{y-3}{-1} = \frac{x-2}{-3}$$

$$\Rightarrow \quad 3y - 9 = x - 2$$

$$\Rightarrow \quad x - 3y + 7 = 0$$

∴ A is the intersection of the line $3x - 4y + 11 = 0$ and $x - 3y + 7 = 0$

$$3x - 4y = -11 \quad \text{... (1)}$$

$$x - 3y = -7 \quad \text{... (2)}$$

$$(1) \Rightarrow \quad 3x - 4y = -11$$

$$(2) \times 3 \Rightarrow \quad 3x - 9y = -21$$

$$\underline{(-) \quad (+) \quad\quad (+)}$$

$$5y = 10$$

$$y = 2$$

Substitute $y = 2$ in (1)

$$3x - 8 = -11$$

$$3x = -3$$

$$x = -1$$

Thus the coordinates of the point A is (–1, 2).

Example 8.56 : Find the foot of the perpendicular from (3, 1) on the line $x + y + 2 = 0$.

Solution : Slope of the line $= -\dfrac{\text{Coefficient of } x}{\text{Coefficient of } y} = -1$

Slope of its perpendicular line = 1

Given point is (3, 1)

$$
\begin{aligned}
\text{Equation of a line is } (y - y_1) &= m\,(x - x_1) \\
y - 1 &= 1\,(x - 3) \\
y - 1 &= x - 3 \\
x - y - 2 &= 0 \\
x - y &= 2 \qquad \ldots (1) \\
x + y &= -2 \qquad \ldots (2) \\
\hline
2x &= 0 \\
x &= 0
\end{aligned}
$$

Substitute $x = 0$ in (2)

$$y = -2$$

$\therefore$ Foot of the perpendicular line is (0, –2).

Example 8.57 : Find the equation of the line through the intersection of lines $3x + 2y = 8$, $5x - 11y + 1 = 0$ and parallel to the line $6x + 13y = 5$.

Solution :

$$
\begin{aligned}
3x + 2y &= 8 \qquad \ldots (1) \\
5x - 11y &= -1 \qquad \ldots (2)
\end{aligned}
$$

Point of intersection of (1) and (2) is (2, 1)

Any line parallel to $6x + 13y - 5 = 0$ is of the form $6x + 13y + k = 0$

This passes through (2, 1)

$$
\begin{aligned}
\therefore\ 12 + 13 + k &= 0 \\
\therefore\ k &= -25
\end{aligned}
$$

$\therefore$ Equation of the required parallel line is $6x + 13y - 25 = 0$.

Example 8.58 : Find the equation of the line through the point of intersection of the lines $5x - 6y = 1$, $3x + 2y + 5 = 0$ and perpendicular to the line $3x - 5y + 11 = 0$

Solution :

$$
\begin{aligned}
5x - 6y &= 1 \qquad \ldots (1) \\
3x + 2y &= -5 \qquad \ldots (2) \\
(1) \Rightarrow \quad 5x - 6y &= 1 \\
(2) \times 3 \Rightarrow \quad 9x + 6y &= -15 \\
\hline
14x &= -14 \\
x &= -1
\end{aligned}
$$

Substitute $x = -1$ in (1)

$$5(-1) - 6y = 1$$

$\therefore$ Point of intersection is $(-1, -1)$

Any line perpendicular to $3x - 5y + 11 = 0$ is of the form

$$5x + 3y + k = 0$$

This passes through $(-1, -1)$

$$-5 - 3 + k = 0$$

$$\therefore k = 8$$

$\therefore$ The required equation is $5x + 3y + 8 = 0$

Example 8.59 : Find the equation of line parallel to Y axis and passing through the point of intersection of the lines $3x - 4y - 9 = 0$ and $x - 4y - 2 = 0$.

Solution : Given lines $3x - 4y - 9 = 0$ ... (1)

$x - 4y - 2 = 0$ ... (2)

By rule of cross multiplication

$$\begin{array}{cccccc} & x & & y & & 1 \\ -4 & & -9 & & 3 & & -4 \\ -4 & & -2 & & 1 & & -4 \end{array}$$

$$\frac{x}{8 - 36} = \frac{y}{-9 + 6} = \frac{1}{-12 + 4}$$

$$\frac{x}{-28} = \frac{y}{-3} = \frac{1}{-8}$$

$$x = \frac{-28}{-8} = \frac{7}{2} \; ; \; y = \frac{-3}{-8} = \frac{3}{8}$$

$\therefore$ Point of intersection is $\left(\frac{7}{2}, \frac{3}{8}\right)$

Equation of line parallel to Y axis is $x = a$.

It passes through $\left(\frac{7}{2}, \frac{3}{8}\right)$ $\quad \therefore a = \frac{7}{2}$

Equation of a line is $x = \frac{7}{2}$

Equation of a line is $2x - 7 = 0$.

EXERCISE - 8.5

1. Find the point of intersection of the lines

   (i) $2x - y + 5 = 0$ and $x + y + 1 = 0$

   (ii) $2x - 3y - 4 = 0$ and $2x + 3y - 8 = 0$

   (iii) $x + y - 5 = 0$ and $3x - y + 1 = 0$

2. Find the equation of the straight line joining the point (2, 3) to the meet of $3x + 4y - 7 = 0$ and $5x - 9y + 4 = 0$.

3. Find the equation of the straight line joining the intersection of the lines $2x - 3y = 10$ and $x + 2y = 12$ and the origin.

4. The lines $2x + 5y - 25 = 0$ and $5x + 4y - 20 = 0$ are diameter of a circle. Find the radius of the circle which passes through the point (3, 4).

5. Find the length of the line segment joining the point (3, 4) and the point of intersection of the lines $5x + y - 6 = 0$ and $2x + y - 3 = 0$.

6. Where does the line $x - 3y + 7 = 0$ meets (i) the X axis, (ii) Y axis.

7. Where do the lines $x - 5 = 0$ and $y = 4$ meet ?

8. Find the length of the line segment which joins the point of intersection of the lines $2x + y - 3 = 0$ and $5x + y - 6 = 0$ and the mid point of the line joining the points (7, 2) and (3, 2).

9. Find the equation of the straight line joining the point of intersection of $3x - y + 9 = 0$ and $2y + x - 4 = 0$ to the point of intersection of $2x + y = 4$ and $2y = x + 3$.

10. A point is collinear with the points (7, 5) and (1, 1). It also lies on the lines $x - 3y + 2 = 0$ find the coordinates of the point.

11. Find the foot of the perpendicular from the origin on the line $3x + 2y = 13$.

12. Find the equation of the line parallel to Y axis and passing through the point of intersection of the lines $3x - 4y - 9 = 0$ and $x - 4y - 2 = 0$.

13. Find the equation of the line through the intersection of $2x + y = 8$, $3x - 7 = y$ and parallel to $4x + y = 11$.

14. Obtain the equation of the straight line passing through the intersection of the lines $x + 3y = 1$ and $x - 2y + 4 = 0$ and parallel to $3x + 4y = 0$.

15. Find the equation of the line passing through the point of intersection of the lines $2x + y - 3 = 0$ and $5x + y - 6 = 0$ and perpendicular to the line joining the points (1, 2) and (2, 1).

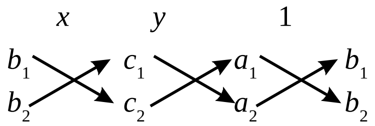
16. Find the point of intersection of $4x + 3y = 10$ and $3x + 5y = 13$. Obtain the equation of the line passing through the point and perpendicular to the latter.

17. Find the equation of the line through the point of intersection of the lines $2x + y - 5 = 0$ and $x + y - 3 = 0$ and bisecting the line segment joining the points (3, –2) and (–5, 6).

8.7 CONCURRENCY OF THREE LINES

Condition that the lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ may be concurrent.

First let us find the point of intersection of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.



$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\therefore\ x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1},\ y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

Hence, the point of intersection is $\left(\dfrac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1},\ \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$

The given lines are concurrent if this point lies on $a_3x + b_3y + c_3 = 0$

$\therefore$ The required condition is

$$a_3\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}\right) + b_3\left(\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right) + c_3 = 0$$

(*i.e.,*) $a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + c_3(a_2b_2 - a_2b_1) = 0$

Example 8.60 : Show that the lines $2x + 3y - 1 = 0$, $3x - y + 2 = 0$ and $7x + 5y = 0$ are concurrent. Find the point of concurrency.

Solution : $2x + 3y - 1 = 0$ ... (1)

$3x - y + 2 = 0$ ... (2)

$$\frac{x}{6-1} = \frac{y}{-3-4} = \frac{1}{-2-9}$$

$$\frac{x}{5} = \frac{1}{-11} \quad \text{and} \quad \frac{y}{-7} = \frac{1}{-11}$$

$$x = \frac{-5}{11} \quad \text{and} \quad y = \frac{7}{11}$$

Point of intersection is $\left(\frac{-5}{11}, \frac{7}{11}\right)$

Substituting $\left(\frac{-5}{11}, \frac{7}{11}\right)$ in $7x + 5y = 0$

$$\text{LHS} = 7\left(\frac{-5}{11}\right) + 5\left(\frac{7}{11}\right)$$

$$= \frac{-35}{11} + \frac{35}{11} = 0 = \text{R.H.S}$$

The equation is satisfied. So $\left(\frac{-5}{11}, \frac{7}{11}\right)$ lies on $7x + 5y = 0$. Hence the lines are concurrent.

Point of concurrency is $\left(\frac{-5}{11}, \frac{7}{11}\right)$

Note : For getting the point of intersection of two straight lines, the rule of cross-multiplication or elimination method can be used.

Example 8.61 : Find the value of a so that $3x + y = 2$, $5x + 2y = 3$ and $ax - y = 3$ are concurrent.

Solution : $3x + y = 2$ ... (1)

$5x + 2y = 3$ ... (2)

(1) × 2 ⇒ $6x + 2y = 4$

(2) ⇒ $5x + 2y = 3$

(–) (–) (–)

Subtracting $x = 1$

sub. $x = 1$ in (1)

$$3(1) + y = 2$$
$$y = 2 - 3$$

Point of intersection (1, –1)

Since the three lines are concurrent, (1, –1) will lie on $ax - y = 3$

$$a(1) - (-1) = 3$$
$$a + 1 = 3$$
$$a = 3 - 1 = 2$$

Example 8.62 : Find the equation of a line passing through the point (3, 4) and concurrent with the lines $x + 2y = 3$ and $2x - y = 1$.

Solution : $x + 2y = 3$

$$2x - y = 1$$

(1) × 1 ⇒ $x + 2y = 3$

(2) × 2 ⇒ $4x - 2y = 2$

$$5x = 5$$
$$x = 1$$

sub. $x = 1$ in (1)

$$1 + 2y = 3$$
$$y = 1$$

point of intersection is (1, 1)

∴ equation of a line joining (1, 1) and (3, 4) is

$$\frac{y-1}{4-1} = \frac{x-1}{3-1}$$

$$\Rightarrow \quad \frac{y-1}{3} = \frac{x-1}{2}$$

$$\Rightarrow \quad 2y - 2 = 3x - 3$$

⇒ $3x - 2y - 1 = 0$ which is the required equation.

Example 8.63 : Find the equation of the line which is concurrent with lines $x - y - 2 = 0$, $3x + 4y + 15 = 0$ and also with the lines $x - 3y + 3 = 0$ and $2x + y = 8$.

Solution : The line concurrent with two pairs of lines means that it passes through the point of intersection of both the pairs.

Now the point of intersection of $x - y - 2 = 0$ and $3x + 4y + 15 = 0$ is given by

$$\frac{x}{-15+8} = \frac{y}{-6-15} = \frac{1}{4+3}$$

$$\frac{x}{-7} = \frac{y}{-21} = \frac{1}{7}$$

$$\frac{x}{-7} = \frac{1}{7} \quad \text{and} \quad \frac{y}{-21} = \frac{1}{7}$$

$$x = -1, \qquad y = -3$$

Point of intersection is (–1, –3)

Also point of intersection of $x - 3y + 3 = 0$ and $2x + y = 8$ is given by

$$\frac{x}{24-3} = \frac{y}{6+8} = \frac{1}{1+6}$$

$$\frac{x}{21} = \frac{y}{14} = \frac{1}{7}$$

$$\frac{x}{21} = \frac{1}{7} \qquad \frac{y}{14} = \frac{1}{7}$$

$$x = 3, \qquad y = 2$$

Point of intersection (3, 2)

Equation of the line joining (–1, –3) and (3, 2) is

$$\frac{y+3}{2+3} = \frac{x+1}{3+1}$$

$$\Rightarrow \quad \frac{y+3}{5} = \frac{x+1}{4}$$

$$\Rightarrow \quad 4y + 12 = 5x + 5$$

$$\Rightarrow \quad 5x - 4y - 7 = 0$$

Equation of a line is $5x - 4y - 7 = 0$

Example 8.64 : Find the equation of the line which is concurrent with the lines $9x + 4y = 1$, $2x - y = 4$ and parallel to the line $3x - y + 7 = 0$.

Solution : Point of intersection of $9x + 4y = 1$ and $2x - y = 4$ is (1, –2) [Verify yourself]

Any line parallel to $3x - y + 7 = 0$ is $3x - y + k = 0$

This line passes through (1, –2)

$$3(1) - (-2) + k = 0$$

$$k = -5$$

Equation of the line is $3x - y - 5 = 0$.

Example 8.65 : Find the equation of the line which is concurrent with the lines $x + 3y - 5 = 0$, $2x - 7y + 16 = 0$ and is perpendicular to the line $x - y + 3 = 0$.

Solution : The point of intersection of the given lines $x + 3y - 5 = 0$ and $2x - 7y + 16 = 0$ is given by

$$\frac{x}{48-35} = \frac{y}{-10-16} = \frac{1}{-7-6}$$

$$\frac{x}{13} = \frac{y}{-26} = \frac{1}{-13}$$

We have, $$\frac{x}{13} = \frac{1}{-13} \quad \text{and} \quad \frac{y}{-26} = \frac{1}{13}$$

$$x = -1, \qquad y = \frac{26}{13}$$

Point of intersection is (–1, 2).

Any equation which is perpendicular to $x - y + 3 = 0$ is of he form $x + y + k = 0$

This passes through (–1, 2)

$$\therefore -1 + 2 + k = 0$$

$$k = -1$$

$\therefore$ The required equation is $x + y - 1 = 0$

Example 8.66 : Obtain the equation of the line which is concurrent with the lines $x - y - 2 = 0$ and $3x + 4y + 15 = 0$ and is perpendicular to the line joining the points (2, 3) and (1, 1).

Solution : Point of intersection of the given lines is (–1, –3)

Slope of the line joining the two points (2, 3) and (1, 1) is $\frac{1-3}{1-2} = \frac{-2}{-1} = 2$

Slope of its perpendicular line is $-\frac{1}{2}$

$$m = -\frac{1}{2}$$

Required line is $(y - y_1) = m\,(x - x_1)$

$$y + 3 = -\frac{1}{2}\ (x + 1)$$

$$2y + 6 = -x - 1$$

$$x + 2y + 7 = 0$$

Equation of a line is $x + 2y + 7 = 0$

(i) Circumcentre, Centroid and Orthocentre

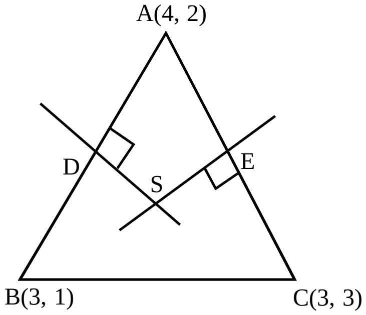
We have learnt that the **perpendicular bisector of the sides of a triangle** are concurrent.

The point of concurrence is called **circumcentre.**

Let us use coordinate geometry to find this point.

Example 8.67 : Find the circumcentre of the triangle whose vertices are A(4, 2), B(3, 1) and C(3, 3).

Solution : Let D and E be the mid-points of the sides AB and AC respectively.



D is $\left(\frac{4+3}{2}, \frac{2+1}{2}\right)$ (i.e.,) $\left(\frac{7}{2}, \frac{3}{2}\right)$

The co-ordinates of E are

$\left(\frac{4+3}{2}, \frac{2+3}{2}\right)$, (i.e.,) $\left(\frac{7}{2}, \frac{5}{2}\right)$

Slope of AB = $\frac{2-1}{4-3} = \frac{1}{1} = 1$

$\therefore$ Slope of the perpendicular bisector of AB is –1.

$\therefore$ Equation of the perpendicular bisector of AB

$$y - \frac{3}{2} = -1\left(x - \frac{7}{2}\right)$$

$$\Rightarrow \quad 2y - 3 = -2x + 7$$

$$\Rightarrow \quad x + y - 5 = 0$$

Slope of the perpendicular bisector of side AC is 1.

$\therefore$ Equation to the perpendicular of AC is $\left(y-\frac{5}{2}\right) = 1\left(x-\frac{7}{2}\right)$

$$\Rightarrow \quad 2y-5 = 2x-7$$

$$\Rightarrow \quad 2x-2y-2 = 0$$

$$\Rightarrow \quad x-y-1 = 0$$

The point of intersection of the perpendicular bisectors of AB and AC.

x	y	1	
1	-5	1	1
-1	-1	1	-1

$$\frac{x}{-1-5} = \frac{y}{-5+1} = \frac{1}{-1-1} \Rightarrow x = 3, y = 2$$

$\therefore$ Point of intersection is (3, 2).

$\therefore$ The circumcentre of the triangle is (3, 2).

(ii) Centroid of a triangle

We have learnt that the medians of a triangle meet at a point. This point is known as centroid.

Using this idea we shall learn how to find the centroid of a triangle.

Example 8.68 : In a Δ ABC, the equation of AB is $x - 2y + 8 = 0$, BC is $x - 3y + 10 = 0$ and CA is $x - y + 2 = 0$ find the centroid of the triangle.

Solution : Let us find the co-ordinates of the vertices A, B and C.

The point of intersection of AB and CA is A.

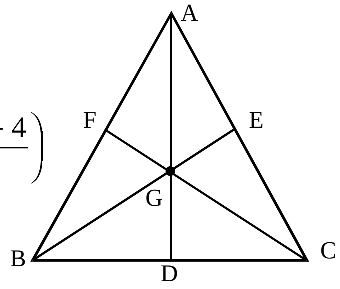
Solving $\quad x - 2y + 8 = 0$

$\quad x - y + 2 = 0 \quad$ we get A as (4, 6)

The point B : Solve for x and y in $x - 2y + 8 = 0$ and $x - 3y + 10 = 0$.

$\therefore$ B is (–4, 2).

Solving $x - 3y + 10 = 0$ and $x - y + 2 = 0$ we get C is (2, 4).



Centroid is $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

$\therefore$ Centroid of ΔABC is $\left(\frac{4-4+2}{3}, \frac{6+2+4}{3}\right)$

$$= \left(\frac{2}{3}, 4\right)$$

(iii) Orthocentre of a Triangle

We have learnt in theoretical geometry, the altitudes of a triangle meet at a point. This point is called **orthocentre.**

Let us find the orthocentre of the given triangle using co-ordinate geometry.

Example 8.69 : Find the orthocentre of the triangle ABC whose vertices are A (–2, –1), B (–1, –4) and C (0, –5).

Solution : Let us find the equation of the altitude AD which is perpendicular to BC.

Slope of BC is $\frac{-4+5}{-1-0} = \frac{1}{-1} = -1$

$\therefore$ Slope of AD is 1.

$\therefore$ Equation of AD is $y + 1 = 1\,(x + 2)$

$$x - y + 1 = 0 \quad \dots (1)$$

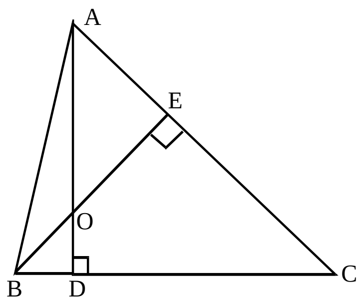
Let us find the equation of BE which is perpendicular to CA.

Slope of CA is $\frac{-5+1}{0+2} = \frac{-4}{2} = -2$

$\therefore$ Slope of BE is $\frac{1}{2}$

Equation of BE is $y + 4 = \frac{1}{2}\,(x + 1)$

$$x - 2y - 7 = 0 \quad \dots (2)$$



The altitudes AD and BE intersect at O. O is the orthocentre. Solving (1) and (2) we get (–9, –8).

$\therefore$ The required orthocentre is (–9, –8).

EXERCISE - 8.6

1. Show that the following set of lines are concurrent. Find their point of concurrency.
   (i) $3x + 4y = 13$, $2x - 7y + 1 = 0$, $5x - y = 14$
   (ii) $3x - 4y + 5 = 0$, $7x - 8y + 5 = 0$, $4x + 5y = 45$
   (iii) $x - 3y + 3 = 0$, $3x + 12y - 12 = 0$, $2x + y = 1$.

2. Find the value of a for which the lines are concurrent.
   (i) $x - 3y + 2 = 0$, $x - 6y + 3 = 0$, $x + ay = 0$
   (ii) $3x - 4y + 5 = 0$, $7x - 8y + 5 = 0$, $4x + ay - 45 = 0$.

3. Show that the straight line $2x + 3y + 19 = 0$ passes through the point of intersection of the straight lines $2x + y + 5 = 0$ and $3x + y + 4 = 0$.

4. Find the equation of the line which passes through (1, 1) and is concurrent with the lines $x + y = 7$ and $2x + y = 16$.

5. Obtain the equation of the line which passes through the origin and is concurrent with the lines $x - y - 4 = 0$ and $7x + y + 20 = 0$.

6. Find the equation of the line which is concurrent with lines $x - y - 2 = 0$, $3x + 4y + 15 = 0$ and also with the lines $x - 3y + 3 = 0$ and $x - 2y + 1 = 0$.

7. Find the equation of the line parallel to the line $x - y - 2 = 0$ and concurrent with the lines $2x + y - 3 = 0$ and $x - 2y + 1 = 0$.

8. Find the equation of the line which is concurrent with the lines $y = x$ and $y = 2 - x$ and is perpendicular to the line $y = 4x + 5$.

9. Find the equation of the line through the point of intersection of the lines $2x - 5y + 1 = 0$ and $3x + 2y - 8 = 0$ and cutting off equal intercepts on the axes.

10. Show that the lines $y = m_1x + c_1$, $y = m_2x + c_2$ and $y = m_3x + c_3$ are concurrent if $m_1 (c_2 - c_3) + m_2 (c_3 - c_1) + m_3 (c_1 - c_2) = 0$.

11. Find the equations of the lines through the intersection of $2x + 3y - 1 = 0$ and $3x - 4y - 6 = 0$ and (i) parallel to the line $5x - y - 2 = 0$ and (ii) perpendicular to the line $x + 2y - 1 = 0$.

12. Find the circumcentre of each of the following triangles whose vertices are
    (i) P (5, 3), Q (4, 4), R (4, 2)
    (ii) P (–2, –2), Q (–6, –2), R (–2, 2)

13. Find the centroid of each of the following triangles whose equations of the sides are :
    (a) $x - 3y + 9 = 0$, $4x + y + 10 = 0$, $5x - 2y - 7 = 0$
    (b) $4x - y - 19 = 0$, $x - y - 4 = 0$, $x + 2y + 11 = 0$

14. Find the orthocentre of each of the following triangles whose vertices are
    (a) (–2, 1), (–1, –4), (0, –5), (b) (–1, –3), (–4, 0), (–7, –3)

15. Find the equation of the line which is concurrent wth the lines $y = x$ and $y = 2 - x$ and is perpendicular to the line $y = 4x + 5$.

9. TRIGONOMETRY

INTRODUCTION

The Science of trigonometry (from Greek words meaning triangle - measure) originally developed as a means of computing unknown sides and angles of triangle and related figures. This aspect of trigonometry was investigated extensively by early Greeks, especially by Hipparachus (180 - 125 B.C) who because of his work in astronomy actually developed spherical rather than plane trigonometry.

Computational trigonometry is still an indispensable tool of the surveyor and developed the scope of trigonometry. The language and methods of trigonometric analysis find a place in nearly every branch of science and engineering. We will find them used extensively in calculus; we will also make use of them when we study wave motion, vibration, alternating current and sound to mention a few areas of applications.

Role of angles in Trigonometry

The concept of angle is one of the most important concepts in geometry. The concepts of equality, sum and difference of angles are important and are used throughout geometry. But the subject of trigonometry is based on the measurement of angles.

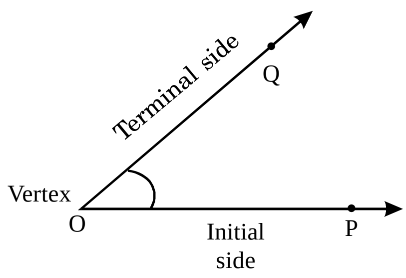
Trigonometry depends on angle measurement and quantities determined by measure of an angle. Trigonometric ratios such as sine, cosine and tangent are used in computations in Trigonometry. These trigonometric ratios relate measurements of angles to measurements of associated straight lines. (i.e., sides of right triangle)

Measurement of Angles

Angles : The figure obtained by rotating a given ray about its end point is called an angle. The original postition of the ray is called the initial side of the angle, whereas the final position of the ray is called the terminal side of the angle and the end point is called the vertex of the angle.

Consider a ray OP. If this ray rotates itself about its end point O and takes the position OQ, then we say that the angle $\angle POQ$ has been generated.

The initial position OP is called the initial side and the final position OQ is called the terminal side of the angle $\angle POQ$. The end point O about which the ray rotates is called the vertex of the angle.



Systems of Measuring angles

There are two commonly used units of measurement of angles.

1. Sexagesimal system
2. Circular system.

1. Sexagesimal System

In this system, the unit of measuring an angle is a degree.

If the circumference of the circle is divided into 360 equal parts, the angle subtended by a part at the centre is one degree. One degree is denoted by 1^{o}.

One degree is divided into 60 equal parts, each part is called a minute. One minute is denoted by 1'

One minute is further divided into 60 equal parts, each part is called a second. One second is denoted by 1''

$\therefore$ 1 degree = 60 minutes [(i.e.,) $1^{o} = 60'$]

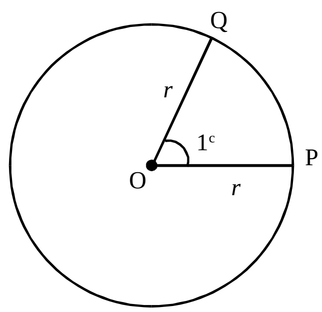
1 minute = 60 seconds [(i.e.,) $1' = 60''$]

2. Circular System

Radian : An angle subtended at the centre of a circle by an arc equal in length to the radius of the circle is called one radian. One radian is denoted by 1^{c}

In the figure, $\angle POQ$ made by the arc PQ which is equal to the radius PQ is one radian.

(i.e.,) $\angle POQ = 1^{c}$ (1 radian)



Relation Between Degree and Radians.

In a circle of radius r units, arc of r units subtends 1^c at the centre. Therefore arc of $2\pi r$ units subtends $2\pi^c$ at the centre.

We know that arc of $2\pi r$ units makes 360^o at the centre.

$$\therefore\ 2\pi \text{ radians} = 360^o$$

$$\pi \text{ radians} = 180^o$$

$$1^c = \left(\frac{180}{\pi}\right)^o$$

$$\text{and } 1^o = \left(\frac{\pi}{180}\right)^c$$

where π is a real constant. To ten places of decimal $\pi = 3.1415926536 \approx \frac{22}{7}$

Note : The value of π is derived as the ratio of the circumference of a circle to its diameter.

Radian measures of some commonly used angles.

Angles in Degrees	0^o	30^o	45^o	60^o	90^o
Angles in Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

Remark (1) : When the angle is measured in the circular measure, then the word 'radian' with the angle is generally omitted.

For example **:** The angle $\frac{\pi}{2}$ radians can also be written as $\left(\frac{\pi}{2}\right)^c$ or simply as $\frac{\pi}{2}$.

Note : 1. For the time being we'll only consider angles between 0^o and 360^o but later, in your higher studies you will learn the angles greater than 360^o and negative angles. (i.e.,) There is neither lower limit nor upper limit for the magnitude of an angle.

Note : 2. Angles will often be designated by Greek letters like α (alpha), β (beta), γ (gamma), θ (theta), ϕ (phi), ψ (chi).

Note : 3. Algebraic operations with angles.

Scalar multiplication : Let θ be the given angle, then scalar multiplication of the angle is $\theta_1 = \lambda\theta$ when λ is any scalar.

Example: (1) $2 \times 30^\circ = 60^\circ$ (2) $2 \times \frac{\pi}{6} = \frac{\pi}{3}$.

We can not add or subtract a scalar from an angle.

Addition (or Subtraction) of two angles : The addition (or subtraction) of two angles θ_1 and θ_2 is given by $\theta = \theta_1 \pm \theta_2$.

Example: (1) $30^\circ + 60^\circ = 90^\circ$ (2) $\frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$.

Review : In class IX, you have learnt some elementary concepts of trigonometry which include trigonometric ratios, finding out other trigonometric ratios if one trigonometric ratio is given, trigonometric ratios of some specific angles.

We would like to repeat the same before we continue further study of trigonometry.

Trigonometric Ratios (T - Ratios) of an acute angle of a right triangle

In XOY-plane, let a revolving line OP starting from OX, trace out $\angle XOP = \theta$.

From P (x, y) draw PM $\perp$ to OX.

In right angled triangle OMP.

OM $= x$ (Adjacent side); PM $= y$ (opposite side); OP $= r$ (hypotenuse).

$$\sin\theta = \frac{\text{Length of opposite side}}{\text{Length of hypotenuse}} = \frac{y}{r}$$

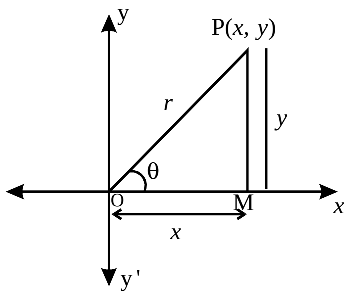
$$\cos\theta = \frac{\text{Length of adjacent side}}{\text{Length of hypotenuse}} = \frac{x}{r}$$

$$\tan\theta = \frac{\text{Length of opposite side}}{\text{Length of adjacent side}} = \frac{y}{x}$$

$$\operatorname{cosec}\theta = \frac{\text{Length of hypotenuse side}}{\text{Length of opposite side}} = \frac{r}{y}$$

$$\sec\theta = \frac{\text{Length of hypotenuse side}}{\text{Length of adjacent side}} = \frac{r}{x}$$

$$\cot\theta = \frac{\text{Length of adjacent side}}{\text{Length of opposite side}} = \frac{x}{y}$$



Reciprocal Relations

$$\sin\theta = \frac{1}{\operatorname{cosec}\theta}\ ; \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta}\ ; \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta}\ ; \qquad \cot\theta = \frac{1}{\tan\theta}$$

Quotient Relations

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \text{and} \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Remark 1 : $\sin\theta$ is read as the "sine of angle θ" and it should never be interpreted as the product of 'sin' and 'θ'

Remark 2 : Notation : $(\sin\theta)^2$ is written as $\sin^2\theta$ (read "sin square θ")

Similarly $(\sin\theta)^n$ is written as $\sin^n\theta$ (read "sin n^{th} power θ"), n being a positive integer.

Caution : $(\sin\theta)^2$ should not be written as $\sin\theta^2$ or as $\sin^2\theta^2$

Remark 3 : Trigonometric ratios depend only on the value of θ and are independent of the lengths of the sides of the right angled triangle.

(i.e.,) Trigonometric ratios are same for the same angle, irrespective of the length of sides.

From the figure

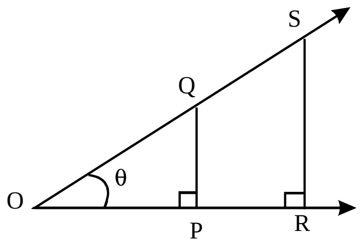
Δ OPQ ||| Δ ORS

Here OP < OR , PQ < RS and OQ < OS

Here the corresponding sides are proportional

$$\Rightarrow \frac{OP}{OR} = \frac{PQ}{RS} = \frac{OQ}{OS}$$

$$\Rightarrow \frac{PQ}{OQ} = \frac{RS}{OS}$$



In right angled Δ OPQ	In right angled Δ ORS
$\sin\theta = \dfrac{PQ}{OQ}$	$\sin\theta = \dfrac{RS}{OS}$

This result shows that the value of sin θ is independent of length of sides.

Trigonometric ratios of Complementary angles.

$\sin(90-\theta) = \cos\theta$, $\cos(90-\theta) = \sin\theta$

$\tan(90-\theta) = \cot\theta$, $\cot(90-\theta) = \tan\theta$

$\sec(90-\theta) = \operatorname{cosec}\theta$, $\operatorname{cosec}(90-\theta) = \sec\theta$.

Trigonometric ratios for angle of measure.

0^o, 30^o, 45^o, 60^o and 90^o in tabular form.

θ	0^o	30^o	45^o	60^o	90^o
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
cot θ	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
sec θ	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
cosec θ	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

9.1 TRIGONOMETRIC IDENTITIES

An equation involving trigonometric ratios of an angle is said to be a trigonometric identity if it is satisfied for all values of θ for which the given trigonometric ratios are defined.

Illustration : $\cos 2\theta = 2\cos^2\theta - 1$ is a trigonometric identity, because it holds for all values of θ. [Verify yourself]

9.1.1 Fundamental Trigonometric Identities

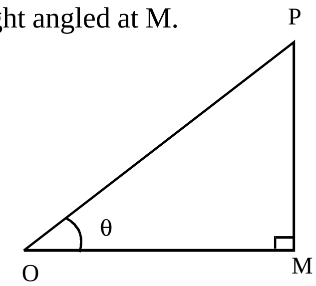
Identity 1. $\sin^2\theta + \cos^2\theta = 1$

Identity 2. $\sec^2\theta - \tan^2\theta = 1$

Identity 3. $\text{cosec}^2\theta - \cot^2\theta = 1$

[We have to prove the above identities for acute angles. But these identities are true for any angle θ in which the trigonometric ratios are defined]

Proof : Let OMP be a right angled triangle right angled at M.



Let $\angle MOP = \theta$

(By Pythagoros theorem)

$OP^2 = OM^2 + PM^2$ ...(1)

$\sin\theta = \dfrac{PM}{OP}$, $\cos\theta = \dfrac{OM}{OP}$

Now $(\sin\theta)^2 + (\cos\theta)^2 = \left(\dfrac{PM}{OP}\right)^2 + \left(\dfrac{OM}{OP}\right)^2$

$\Rightarrow \quad \sin^2\theta + \cos^2\theta = \dfrac{(PM)^2 + (OM)^2}{(OP)^2}$

$\Rightarrow \quad \sin^2\theta + \cos^2\theta = \dfrac{OP^2}{OP^2} = 1$ [from (1)]

(2) To prove that $\sec^2\theta - \tan^2\theta = 1$

From the right angled Δ OMP

$\tan\theta = \dfrac{PM}{OM}$, $\sec\theta = \dfrac{OP}{OM}$

$\sec^2\theta - \tan^2\theta = \left(\dfrac{OP}{OM}\right)^2 - \left(\dfrac{PM}{OM}\right)^2$

$\Rightarrow \quad \sec^2\theta - \tan^2\theta = \dfrac{OP^2 - PM^2}{OM^2}$

$= \dfrac{OM^2}{OM^2} = 1$ [from (1) $OP^2 = OM^2 + PM^2 \Rightarrow OP^2 - PM^2 = OM^2$].

$$\therefore \sec^2\theta - \tan^2\theta = 1.$$

(3) To prove $\operatorname{cosec}^2\theta - \cot^2\theta = 1.$

From the right angled Δ OMP

$\operatorname{cosec}\theta = \dfrac{OP}{PM}$, $\cot\theta = \dfrac{OM}{PM}$

Now $(\operatorname{cosec}\theta)^2 - (\cot\theta)^2 = \left(\dfrac{OP}{PM}\right)^2 - \left(\dfrac{OM}{PM}\right)^2$

$\Rightarrow \quad \operatorname{cosec}^2\theta - \cot^2\theta = \dfrac{OP^2 - OM^2}{PM^2}$

$= \dfrac{PM^2}{PM^2}$ [from (1) $OP^2 = OM^2 + PM^2$]

$\therefore \operatorname{cosec}^2\theta - \cot^2\theta = 1$

Corollary (1) : From the identity, $\sin^2\theta + \cos^2\theta = 1$, we deduce that

$\sin^2\theta = 1 - \cos^2\theta$ and $\cos^2\theta = 1 - \sin^2\theta$.

Corollary (2) : From the identity, $\sec^2\theta - \tan^2\theta = 1$, we deduce that

$\sec^2\theta = 1 + \tan^2\theta$ and $\tan^2\theta = \sec^2\theta - 1$.

Corollary (3) : From the identity, $\operatorname{cosec}^2\theta - \cot^2\theta = 1$, we deduce that

$\operatorname{cosec}^2\theta = 1 + \cot^2\theta$ and $\cot^2\theta = \operatorname{cosec}^2\theta - 1$.

Example 9.1 : Simplify (i) $\sin\theta(\operatorname{cosec}\theta - \sin\theta)$

(ii) $(\sec\theta + \tan\theta)(1 - \sin\theta)$

(iii) $\dfrac{1+\tan^2\theta}{1+\cot^2\theta}$

Solution : (i) $\sin\theta(\operatorname{cosec}\theta - \sin\theta)$

$$= \sin\theta\left(\frac{1}{\sin\theta} - \sin\theta\right) = \sin\theta\left(\frac{1-\sin^2\theta}{\sin\theta}\right)$$

$= 1 - \sin^2 \theta$ [since $1 - \sin^2\theta = \cos^2 \theta$]

$= \cos^2 \theta$

(ii) $(\sec \theta + \tan \theta)(1 - \sin \theta)$

$$= \left(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)(1 - \sin\theta) = \left(\frac{1+\sin\theta}{\cos\theta}\right)(1 - \sin\theta)$$

$$= \frac{1-\sin^2\theta}{\cos\theta}$$ [since $(a + b)(a - b) = a^2 - b^2$]

$$= \frac{\cos^2\theta}{\cos\theta}$$ [since $1 - \sin^2 \theta = \cos^2 \theta$]

$= \cos \theta$

(iii) $$\frac{1+\tan^2\theta}{1+\cot^2\theta} = \frac{\sec^2\theta}{\operatorname{cosec}^2\theta}$$

$$= \frac{\frac{1}{\cos^2\theta}}{\frac{1}{\sin^2\theta}} = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta.$$

Example 9.2 : Prove that $\tan^4 \phi + \tan^2 \phi = \sec^4 \phi - \sec^2 \phi$, for $0^\circ \le \theta < 90^\circ$

Solution :

$\tan^4\phi + \tan^2 \phi = \tan^2 \phi (\tan^2\phi + 1)$

$= \tan^2\phi (\sec^2 \phi) = (\sec^2\phi - 1)(\sec^2 \phi)$

$= \sec^4\phi - \sec^2\phi.$

Example 9.3 : Prove the following identities :

(i) $\cos^2 \theta + \dfrac{1}{1+\cot^2\theta} = 1$

(ii) $\sec \theta (1 - \sin \theta)(\sec \theta + \tan \theta) = 1$

Solution :

(i) $\cos^2 \theta + \dfrac{1}{1+\cot^2\theta} = \cos^2 \theta + \dfrac{1}{\operatorname{cosec}^2\theta}$ [since $1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$]

$= \cos^2 \theta + \sin^2 \theta = 1$ [since $\sin^2 \theta + \cos^2 \theta = 1$]

(ii) sec θ (1 – sin θ) (sec θ + tan θ)

$$= \frac{1}{\cos\theta}\,(1-\sin\theta)\left(\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}\right)$$

$$= \left(\frac{1-\sin\theta}{\cos\theta}\right)\left(\frac{1+\sin\theta}{\cos\theta}\right)$$

$$= \frac{1^2-\sin^2\theta}{\cos^2\theta} = \frac{\cos^2\theta}{\cos^2\theta} = 1.$$

Example 9.4 : Prove the following identities.

(i) $\sqrt{\frac{\sec\theta+1}{\sec\theta-1}} = \frac{\sec\theta+1}{\tan\theta} = \frac{\sin\theta}{1-\cos\theta}$

(ii) $\sqrt{\cot^2\theta-\cos^2\theta} = \cot\theta\cos\theta$

(iii) $\sqrt{\frac{\sec\theta-1}{\sec\theta+1}} + \sqrt{\frac{\sec\theta+1}{\sec\theta-1}} = 2\,\text{cosec}\,\theta.$

Solution : (i) $\sqrt{\frac{\sec\theta+1}{\sec\theta-1}} = \sqrt{\frac{\sec\theta+1}{\sec\theta-1}\times\frac{\sec\theta+1}{\sec\theta+1}}$

$$= \sqrt{\frac{(\sec\theta+1)^2}{\sec^2\theta-1}}$$

$$= \sqrt{\frac{(\sec\theta+1)^2}{\tan^2\theta}} \qquad [\text{since } \sec^2\theta - 1 = \tan^2\theta]$$

$$\sqrt{\frac{\sec\theta+1}{\sec\theta-1}} = \frac{\sec\theta+1}{\tan\theta} \qquad \text{..........}\ (1)$$

$$= \frac{\frac{1}{\cos\theta}+1}{\frac{\sin\theta}{\cos\theta}} = \frac{1+\cos\theta}{\sin\theta}\times\frac{1-\cos\theta}{1-\cos\theta}$$

$$= \frac{1^2 - \cos^2\theta}{\sin\theta\,(1-\cos\theta)}$$

$$\therefore \sqrt{\frac{\sec\theta+1}{\sec\theta-1}} = \frac{\sin^2\theta}{\sin\theta\,(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta} \qquad \text{... (2)}$$

From (1) and (2) we get

$$\sqrt{\frac{\sec\theta+1}{\sec\theta-1}} = \frac{\sin\theta}{1-\cos\theta} = \frac{\sec\theta+1}{\tan\theta}$$

(ii) $$\sqrt{\cot^2\theta - \cos^2\theta} = \sqrt{\frac{\cos^2\theta}{\sin^2\theta} - \cos^2\theta}$$

$$= \sqrt{\cos^2\theta\left(\frac{1}{\sin^2\theta} - 1\right)} = \sqrt{\cos^2\theta\,\frac{(1-\sin^2\theta)}{\sin^2\theta}}$$

$$= \sqrt{\frac{\cos^2\theta.\cos^2\theta}{\sin^2\theta}} \qquad [\text{since } 1 - \sin^2\theta = \cos^2\theta]$$

$$= \sqrt{\cot^2\theta\,\cos^2\theta} \qquad \left[\text{since } \frac{\cos^2\theta}{\sin^2\theta} = \cot^2\theta\right]$$

$$\therefore \sqrt{\cot^2\theta - \cos^2\theta} = \cot\theta\,\cos\theta$$

(iii) $$\sqrt{\frac{\sec\theta-1}{\sec\theta+1}} + \sqrt{\frac{\sec\theta+1}{\sec\theta-1}}$$

$$= \sqrt{\frac{\sec\theta-1}{\sec\theta+1}\times\frac{\sec\theta-1}{\sec\theta-1}} + \sqrt{\frac{\sec\theta+1}{\sec\theta-1}\times\frac{\sec\theta+1}{\sec\theta+1}}$$

$$= \sqrt{\frac{(\sec\theta-1)^2}{\sec^2\theta-1}} + \sqrt{\frac{(\sec\theta+1)^2}{\sec^2\theta-1}}$$

$$= \sqrt{\frac{(\sec\theta-1)^2}{\tan^2\theta}} + \sqrt{\frac{(\sec\theta+1)^2}{\tan^2\theta}}$$

$$= \frac{\sec\theta - 1}{\tan\theta} + \frac{\sec\theta + 1}{\tan\theta}$$

$$= \frac{\sec\theta - 1 + \sec\theta + 1}{\tan\theta} = \frac{2\sec\theta}{\tan\theta} = \frac{\frac{2}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}$$

$$= \frac{2}{\sin\theta} = 2 \text{ cosec } \theta.$$

Example 9.5 : Prove the following trigonometric identities.

(i) $\frac{2\tan\phi}{1+\tan^2\phi} = 2 \sin\phi \cos\phi$

(ii) $\frac{1+\sin\phi}{\cos\phi} + \frac{\cos\phi}{1+\sin\phi} = 2 \sec\phi$

(iii) $\frac{\cos\phi}{1-\tan\phi} + \frac{\sin\phi}{1-\cot\phi} = \cos\phi + \sin\phi$

Solution : (i) $\frac{2\tan\phi}{1+\tan^2\phi} = \frac{\frac{2\sin\phi}{\cos\phi}}{1+\frac{\sin^2\phi}{\cos^2\phi}}$ [since $\tan\phi = \frac{\sin\phi}{\cos\phi}$]

$$= \frac{\frac{2\sin\phi}{\cos\phi}}{\frac{\cos^2\phi + \sin^2\phi}{\cos^2\phi}}$$

$\therefore \frac{2\tan\phi}{1+\tan^2\phi} = \frac{\frac{2\sin\phi}{\cos\phi}}{\frac{1}{\cos^2\phi}}$ [$\sin^2\phi + \cos^2\phi = 1$]

$$= 2 \sin\phi \cos\phi$$

(ii) $\dfrac{1+\sin\phi}{\cos\phi} + \dfrac{\cos\phi}{1+\sin\phi}$

$$= \frac{(1+\sin\phi)^2 + \cos^2\phi}{\cos\phi\,(1+\sin\phi)}$$

$$= \frac{1+\sin^2\phi + 2\sin\phi + \cos^2\phi}{\cos\phi\,(1+\sin\phi)}$$

$$= \frac{2+2\sin\phi}{\cos\phi\,(1+\sin\phi)} \quad [\text{since } \sin^2\phi + \cos^2\phi = 1]$$

$$= \frac{2\,(1+\sin\phi)}{\cos\phi\,(1+\sin\phi)} = \frac{2}{\cos\phi}$$

$$= 2\sec\phi$$

(iii) $\dfrac{\cos\phi}{1-\tan\phi} + \dfrac{\sin\phi}{1-\cot\phi} = \dfrac{\cos\phi}{1-\dfrac{\sin\phi}{\cos\phi}} + \dfrac{\sin\phi}{1-\dfrac{\cos\phi}{\sin\phi}}$

$$= \frac{\cos\phi}{\dfrac{\cos\phi-\sin\phi}{\cos\phi}} + \frac{\sin\phi}{\dfrac{\sin\phi-\cos\phi}{\sin\phi}}$$

$$= \frac{\cos^2\phi}{\cos\phi-\sin\phi} + \frac{\sin^2\phi}{\sin\phi-\cos\phi}$$

$$= \frac{\cos^2\phi}{\cos\phi-\sin\phi} - \frac{\sin^2\phi}{\cos\phi-\sin\phi}$$

$$= \frac{\cos^2\phi-\sin^2\phi}{\cos\phi-\sin\phi}$$

$$= \frac{(\cos\phi+\sin\phi)\,(\cos\phi-\sin\phi)}{(\cos\phi-\sin\phi)}$$

$$= \cos\phi + \sin\phi$$

Example 9.6 : Prove the following identities.

(i) $$\frac{\tan\theta+\sin\theta}{\tan\theta-\sin\theta} = \frac{\sec\theta+1}{\sec\theta-1}$$

(ii) $$\frac{\sin\theta-2\sin^3\theta}{2\cos^3\theta-\cos\theta} = \tan\theta$$

(iii) $$\frac{\sin\theta+\cos\theta}{\sin\theta-\cos\theta} + \frac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta} = \frac{2}{2\sin^2\theta-1} = \frac{2}{1-2\cos^2\theta}$$

(iv) $$\frac{(1+\sin\theta)^2+(1-\sin\theta)^2}{\cos^2\theta} = 2\left(\frac{1+\sin^2\theta}{1-\sin^2\theta}\right)$$

Solution : $$\frac{\tan\theta+\sin\theta}{\tan\theta-\sin\theta} = \frac{\dfrac{\sin\theta}{\cos\theta}+\dfrac{\sin\theta}{1}}{\dfrac{\sin\theta}{\cos\theta}-\sin\theta}$$

$$= \frac{\sin\theta+\sin\theta\cos\theta}{\sin\theta-\sin\theta\cos\theta}$$

$$= \frac{\sin\theta\,(1+\cos\theta)}{\sin\theta\,(1-\cos\theta)}$$

$$= \frac{1+\cos\theta}{1-\cos\theta} = \frac{\sec\theta+1}{\sec\theta-1}$$

(ii) $$\frac{\sin\theta-2\sin^3\theta}{2\cos^3\theta-\cos\theta} = \frac{\sin\theta\,(1-2\sin^2\theta)}{\cos\theta\,(2\cos^2\theta-1)}$$

$$= \tan\theta\left[\frac{1-2\sin^2\theta}{2\,(1-\sin^2\theta)-1}\right]$$

$$= \tan\theta\left[\frac{1-2\sin^2\theta}{1-2\sin^2\theta}\right] = \tan\theta.$$

(iii) $\dfrac{\sin\theta+\cos\theta}{\sin\theta-\cos\theta}+\dfrac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta}$

$$=\frac{(\sin\theta+\cos\theta)^2+(\sin\theta-\cos\theta)^2}{(\sin\theta-\cos\theta)(\sin\theta+\cos\theta)}$$

$$=\frac{(\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta)+(\sin^2\theta+\cos^2\theta-2\sin\theta\cos\theta)}{(\sin^2\theta-\cos^2\theta)}$$

$$=\frac{1+2\sin\theta\cos\theta+1-2\sin\theta\cos\theta}{\sin^2\theta-\cos^2\theta}$$

$$=\frac{2}{\sin^2\theta-\cos^2\theta}\qquad\text{... (1)}$$

$$=\frac{2}{\sin^2\theta-(1-\sin^2\theta)}=\frac{2}{2\sin^2\theta-1}$$

From (1) $\dfrac{\sin\theta+\cos\theta}{\sin\theta-\cos\theta}+\dfrac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta}=\dfrac{2}{\sin^2\theta-\cos^2\theta}$

$$=\frac{2}{1-\cos^2\theta-\cos^2\theta}=\frac{2}{1-2\cos^2\theta}$$

(iv) $\dfrac{(1+\sin\theta)^2+(1-\sin\theta)^2}{\cos^2\theta}=\dfrac{2\,(1+\sin^2\theta)}{1-\sin^2\theta}$

[since $(a+b)^2+(a-b)^2=2\,(a^2+b^2)$]

$$=2\left(\frac{1+\sin^2\theta}{1-\sin^2\theta}\right)$$

Example 9.7 : Prove that

$$\frac{1}{\operatorname{cosec}A-\cot A}-\frac{1}{\sin A}=\frac{1}{\sin A}-\frac{1}{\operatorname{cosec}A+\cot A}$$

Solution :

$$\text{L.H.S.} = \frac{1}{\operatorname{cosec} A - \cot A} - \frac{1}{\sin A} = \frac{1}{\operatorname{cosec} A - \cot A} \times \frac{\operatorname{cosec} A + \cot A}{\operatorname{cosec} A + \cot A} - \frac{1}{\sin A}$$

$$= \frac{\operatorname{cosec} A + \cot A}{\operatorname{cosec}^2 A - \cot^2 A} - \operatorname{cosec} A.$$

$$= \operatorname{cosec} A + \cot A - \operatorname{cosec} A$$

$$\therefore \frac{1}{\operatorname{cosec} A - \cot A} - \frac{1}{\sin A} = \cot A \qquad \text{... (1)}$$

$$\text{RHS} = \frac{1}{\sin A} - \frac{1}{\operatorname{cosec} A + \cot A}$$

$$= \frac{1}{\sin A} - \frac{1}{\operatorname{cosec} A + \cot A} \times \frac{\operatorname{cosec} A - \cot A}{\operatorname{cosec} A - \cot A}$$

$$= \operatorname{cosec} A - \left[\frac{\operatorname{cosec} A - \cot A}{\operatorname{cosec}^2 A - \cot^2 A}\right]$$

$$= \operatorname{cosec} A - \operatorname{cosec} A + \cot A.$$

$$\frac{1}{\sin A} - \frac{1}{\operatorname{cosec} A + \cot A} = \cot A \qquad \text{... (2)}$$

From (1) and (2), $\frac{1}{\operatorname{cosec} A - \cot A} - \frac{1}{\sin A} = \frac{1}{\sin A} - \frac{1}{\operatorname{cosec} A + \cot A}$

Example 9.8 : Show that

(i) $\sin^4 \theta + \cos^4 \theta - 2\sin^2 \theta \cos^2 \theta = 1 - 4 \sin^2 \theta \cos^2 \theta$

(ii) $2 (\sin^6 \theta + \cos^6 \theta) - 3 (\sin^4 \theta + \cos^4 \theta) + 1 = 0$

(iii) $\sec^6 \theta = \tan^6 \theta + 3 \tan^2 \theta \sec^2 \theta + 1$

Solution :

(i) $\sin^4 \theta + \cos^4 \theta - 2\sin^2 \theta \cos^2 \theta$

$= \sin^2\theta \sin^2\theta + \cos^2\theta \cos^2\theta - 2\sin^2\theta \cos^2\theta$

$= \sin^2\theta (1 - \cos^2\theta) + \cos^2\theta (1 - \sin^2\theta) - 2\sin^2\theta \cos^2\theta$

$= \sin^2\theta - \sin^2\theta \cos^2\theta + \cos^2\theta - \sin^2\theta \cos^2\theta - 2 \sin^2\theta \cos^2\theta$

$= \sin^2\theta + \cos^2\theta - 4 \sin^2\theta \cos^2\theta$

$= 1 - 4 \sin^2\theta \cos^2\theta$

(ii) $2 (\sin^6\theta + \cos^6\theta) - 3 (\sin^4\theta + \cos^4\theta) + 1 = 0$

Consider $\sin^6\theta + \cos^6\theta = (\sin^2\theta)^3 + (\cos^2\theta)^3$

$= (\sin^2\theta + \cos^2\theta)^3 - 3 \sin^2\theta \cos^2\theta (\sin^2\theta + \cos^2\theta)$

[since $a^3 + b^3 = (a + b)^3 - 3ab\,(a + b)$]

$= 1 - 3 \sin^2\theta \cos^2\theta$ [since $\sin^2\theta + \cos^2\theta = 1$]

Now $\sin^4\theta + \cos^4\theta = (\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta \cos^2\theta$

[since $a^2 + b^2 = (a + b)^2 - 2ab$]

$= 1 - 2 \sin^2\theta \cos^2\theta$

$\therefore\ 2 (\sin^6\theta + \cos^6\theta) - 3 (\sin^4\theta + \cos^4\theta) + 1$

$= 2 [1 - 3 \sin^2\theta \cos^2\theta] - 3 [1 - 2 \sin^2\theta \cos^2\theta] + 1$

$= 2 - 6 \sin^2\theta \cos^2\theta - 3 + 6 \sin^2\theta \cos^2\theta + 1$

$= 0.$

(iii) $\sec^6\theta = [\sec^2\theta]^3 = [1 + \tan^2\theta]^3$

$= 1 + \tan^6\theta + 3 \tan^2\theta + 3 \tan^4\theta$

$= 1 + \tan^6\theta + 3 \tan^2\theta (1 + \tan^2\theta)$

$= 1 + \tan^6\theta + 3 \tan^2\theta \sec^2\theta$

$\therefore\ \sec^6\theta = \tan^6\theta + 3 \tan^2\theta \sec^2\theta + 1$

Example 9.9 : (i) If $A + B = 90^o$ then prove that

$$\sqrt{\frac{\tan A \tan B + \tan A \cot B}{\sin A \sec B} - \frac{\sin^2 B}{\cos^2 A}} = \tan A.$$

(ii) If A, B, C are the interior angles of a triangle, prove that

$$\tan\left(\frac{B + C}{2}\right) = \cot\left(\frac{A}{2}\right)$$

Solution : Here A + B = 90° $\Rightarrow$ B = 90° – A

(i) $\sqrt{\dfrac{\tan A \tan B + \tan A \cot B}{\sin A \sec B} - \dfrac{\sin^2 B}{\cos^2 A}}$

$$= \sqrt{\frac{\tan A \tan (90 - A) + \tan A \cot (90 - A)}{\sin A \sec (90 - A)} - \frac{\sin^2 (90 - A)}{\cos^2 A}}$$

$$= \sqrt{\frac{\tan A \cot A + \tan A \tan A}{\sin A \operatorname{cosec} A} - \frac{\cos^2 A}{\cos^2 A}}$$

$$= \sqrt{\frac{1 + \tan^2 A}{1} - 1}$$ [since tan A cot A = tan A. $\dfrac{1}{\tan A}$ = 1

|||ly sin A cosec A = 1]

$$= \sqrt{\tan^2 A} = \tan A.$$

(iii) Here A + B + C = 180°

$\Rightarrow$ B + C = 180° – A

$$\Rightarrow \frac{B + C}{2} = 90^\circ - \frac{A}{2}$$

$$\Rightarrow \tan\left(\frac{B + C}{2}\right) = \tan\left(90^\circ - \frac{A}{2}\right)$$

$$\Rightarrow \tan\left(\frac{B + C}{2}\right) = \cot\left(\frac{A}{2}\right)$$ [since tan (90° – θ) = cot θ]

Conditional Identities

Identities which are true under certain conditions are called conditional identities.

Example 9.10 : (i) If $\dfrac{\cos\alpha}{\cos\beta} = m$ and $\dfrac{\cos\alpha}{\sin\beta} = n$ show that $(m^2 + n^2)\cos^2\beta = n^2$.

(ii) If $x = r \sin A \cos B$, $y = r \sin A \sin B$ and $z = r \cos A$ prove that $r^2 = x^2 + y^2 + z^2$.

(iii) If $x = a \sec\theta + b \tan\theta$, $y = a \tan\theta + b \sec\theta$ prove that $x^2 - y^2 = a^2 - b^2$.

Solution : (i) $(m^2 + n^2) \cos^2 \beta = \left[\frac{\cos^2\alpha}{\cos^2\beta} + \frac{\cos^2\alpha}{\sin^2\beta}\right] \cos^2\beta$

$$= \left[\frac{\cos^2\alpha \sin^2\beta + \cos^2\alpha \cos^2\beta}{\sin^2\beta\cos^2\beta}\right] \cos^2\beta$$

$$= \cos^2\alpha \left[\frac{\sin^2\beta + \cos^2\beta}{\sin^2\beta\cos^2\beta}\right] \cos^2\beta$$

$$= \frac{\cos^2\alpha}{\sin^2\beta} = \left(\frac{\cos\alpha}{\sin\beta}\right)^2 = n^2.$$

(ii) $x^2 + y^2 + z^2 = [r \sin A \cos B]^2 + [r \sin A \sin B]^2 + [r \cos A]^2$

$= r^2 \sin^2 A \cos^2 B + r^2 \sin^2 A \sin^2 B + r^2 \cos^2 A$

$= r^2 \sin^2 A [\cos^2 B + \sin^2 B] + r^2 \cos^2 A$

$= r^2 [\sin^2 A + \cos^2 A]$ [since $\cos^2 B + \sin^2 B = 1$]

$= r^2$

(iii) $x^2 - y^2 = [a \sec\theta + b \tan\theta]^2 - [a \tan\theta + b \sec\theta]^2$

$= a^2 \sec^2\theta + b^2 \tan^2\theta + 2ab \sec\theta \tan\theta$
$- [a^2 \tan^2\theta + b^2 \sec^2\theta + 2ab \sec\theta \tan\theta]$

$= a^2 (\sec^2\theta - \tan^2\theta) - b^2 (\sec^2\theta - \tan^2\theta)$

$= a^2 - b^2$ [since $\sec^2\theta - \tan^2\theta = 1$]

EXERCISE - 9.1

1. Prove the following identities.

(i) $\frac{\sin^4\theta - \cos^4\theta}{\sin^2\theta - \cos^2\theta} = 1$ (ii) $(1 + \cot^2\theta)(1 - \cos\theta)(1 + \cos\theta) = 1$

(iii) $\sin^2\theta + \frac{1}{1+\tan^2\theta} = 1$ (iv) $\cos^2\theta + \frac{1}{1+\cot^2\theta} = 1$

(v) $(\operatorname{cosec}\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta) = 1$

(vi) $(\sec^2\theta - 1)(\operatorname{cosec}^2\theta - 1) = 1$

2. (i) $1 + \frac{\cot^2\theta}{1+\text{cosec}\,\theta} = \text{cosec}\,\theta$ (ii) $\frac{1+\cos\theta-\sin^2\theta}{\sin\theta\,(1+\cos\theta)} = \cot\theta$

(iii) $\frac{(1+\tan^2\theta)(\cot\theta)}{\text{cosec}^2\theta} = \tan\theta$ (iv) $\frac{\tan^2\phi}{\sec\phi+1} = \sec\phi - 1$

(v) $\frac{\tan^3\phi - 1}{\tan\phi - 1} = \sec^2\phi + \tan\phi$

3. (i) $\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} + \sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = 2\sec\theta$ (ii) $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} + \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = 2\,\text{cosec}\,\theta$

(iii) $\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = 2\,\text{cosec}\,\theta$ (iv) $\frac{\sin\theta}{1-\cos\theta} + \frac{\tan\theta}{1+\cos\theta} = \sec\theta\,\text{cosec}\,\theta + \cot\theta$

(v) $\frac{\sin\theta}{\cot\theta+\text{cosec}\,\theta} = 2 + \frac{\sin\theta}{\cot\theta-\text{cosec}\,\theta}$

(vi) $\frac{\sec\theta-\tan\theta}{\sec\theta+\tan\theta} = 1 - 2\sec\theta\tan\theta + 2\tan^2\theta$

(vii) $\frac{\cos\theta}{1-\tan\theta} + \frac{\sin\theta}{1-\cot\theta} = \cos\theta + \sin\theta$

(viii) $\frac{\tan\theta}{1-\cot\theta} + \frac{\cot\theta}{1-\tan\theta} = 1 + \tan\theta + \cot\theta = 1 + \sec\theta\,\text{cosec}\,\theta$

4. (i) $\frac{1}{\sec A+\tan A} - \frac{1}{\cos A} = \frac{1}{\cos A} - \frac{1}{\sec A-\tan A}$

(ii) $\frac{(1+\cot A+\tan A)(\sin A-\cos A)}{\sec^3 A-\text{cosec}^3 A} = \sin^2 A\cos^2 A.$

5. (i) $\cos^4 A - \cos^2 A = \sin^4 A - \sin^2 A$

(ii) $\text{cosec}^6\theta = \cot^6\theta + 3\cot^2\theta\,\text{cosec}^2\theta + 1.$

(iii) $\sin^8\theta - \cos^8\theta = (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta\cos^2\theta)$

6. (i) $(\sin\theta + \sec\theta)^2 + (\cos\theta + \text{cosec}\,\theta)^2 = (1 + \sec\theta\,\text{cosec}\,\theta)^2$

(ii) $(1 + \tan\alpha\tan\beta)^2 + (\tan\alpha - \tan\beta)^2 = \sec^2\alpha\sec^2\beta$

(iii) $(\tan\alpha + \text{cosec}\,\beta)^2 - (\cot\beta - \sec\alpha)^2 = 2\tan\alpha\cot\beta\,(\text{cosec}\,\alpha + \sec\beta)$

7. (i) If $\cot\theta + \tan\theta = x$ and $\sec\theta - \cos\theta = y$ prove that $(x^2 y)^{2/3} - (xy^2)^{2/3} = 1$.

(ii) If $\sin\theta + \sin^2\theta = 1$, P.T. $\cos^2\theta + \cos^4\theta = 1$.

(iii) If $\sin\theta + \sin^2\theta = 1$, find the value of

$\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta + 2\cos^4\theta + 2\cos^2\theta - 2$.

(iv) If $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$, and $\frac{x}{a}\sin\theta - \frac{y}{b}\cos\theta = 1$. P.T. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$.

9.2 Trigonometric table for standard angles [0^o, 30^o, 45^o, 60^o, 90^o]

In class IX, you have already learnt about the values of trigonometric ratios of 0^o, 30^o, 45^o, 60^o and 90^o. These values are also given in section 9.1 for ready reference. In this section we shall be using these values for evaluating expressions involving trigonometric ratios as illustrated in the following examples.

Example 9.11 : If A= 30^o , verify the following (i) $\sin^2 A + \cos^2 A = 1$

(ii) $\text{Sec}^2 A - \tan^2 A = 1$, (iii) $\text{cosec}^2 A - \cot^2 A = 1$.

Solution :(i) $\sin^2 A + \cos^2 A = \sin^2 30^o + \cos^2 30^o$

$$= (\sin 30^o)^2 + (\cos 30^o)^2$$

$$= \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$$

(ii) $\sec^2 A - \tan^2 A = \sec^2 30^o - \tan^2 30^o$

$$= (\sec 30^o)^2 - (\tan 30^o)^2$$

$$= \left(\frac{2}{\sqrt{3}}\right)^2 - \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{4}{3} - \frac{1}{3} = \frac{3}{3} = 1.$$

(iii) $\text{cosec}^2 A - \cot^2 A = \text{cosec}^2 30^o - \cot^2 30^o$.

$$= 2^2 - (\sqrt{3})^2 = 4 - 3 = 1.$$

Example 9.12 : If A = 30^o verify that

(i) $\sin 2A = 2\sin A\cos A$. (ii) $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$ (iii) $\tan 2A = \frac{2\tan A}{1+\tan^2 A}$

Solution : (i) LHS = $\sin 2A = \sin 2(30^o) = \sin 60^o$

$$= \frac{\sqrt{3}}{2} \qquad .. (1)$$

RHS = $2 \sin A \cos A = 2 \sin 30^o \cos 30^o = 2 \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$

$$= \frac{\sqrt{3}}{2} \quad \text{... (1)}$$

From (1) & (2), $\sin 2A = 2 \sin A \cos A$.

(ii) LHS = $\cos 2A = \cos 2(30^o) = \cos 60^o$

$$= \frac{1}{2} \quad \text{... (1)}$$

$$\text{RHS} = \frac{1-\tan^2 A}{1+\tan^2 A} = \frac{1-\tan^2 30^o}{1+\tan^2 30^o} = \frac{1-\left(\frac{1}{\sqrt{3}}\right)^2}{1+\left(\frac{1}{\sqrt{3}}\right)^2}$$

$$= \frac{1}{2} \quad \text{... (2)}$$

From (1) & (2) $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$

(iii) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

LHS = $\tan(2A) = \tan 2(30^o) = \tan 60^o$

$$= \sqrt{3} \quad \text{... (1)}$$

$$\text{RHS} = \frac{2\tan A}{1-\tan^2 A} = \frac{2\tan 30^o}{1-\tan^2 30^o}$$

$$= \left(\frac{\frac{2}{\sqrt{3}}}{1-\frac{1}{3}}\right) = \sqrt{3} \quad \text{... (2)}$$

From (1) and (2) $\tan(2A) = \frac{2\tan A}{1-\tan^2 A}$

Example 9.13 : If A = 30°, B = 60° verify that

(i) sin (A+B) = sin A cos B + cos A sin B

(ii) cos (A+B) = cos A cos B – sin A sin B

Solution : (i)

LHS = sin (A + B) = sin (30° + 60°) = sin 90° = 1 ... (1)

RHS = sin A cos B + cos A sin B = sin 30° cos 60° + cos 30° sin 60°

$$= \frac{1}{2}\left(\frac{1}{2}\right) + \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{1}{4} + \frac{3}{4} = 1 \qquad \text{... (2)}$$

from (1) & (2) sin (A + B) = sin A cos B + cos A sin B

(ii) LHS = cos (A+B) = cos (30° + 60°) = cos 90° = 0 ... (1)

RHS = cos A cos B – sin A Sin B = cos 30° cos 60° – sin 30° sin 60°

$$= \frac{\sqrt{3}}{2}\left(\frac{1}{2}\right) - \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}$$

= 0 ... (2)

From (1) & (2), cos (A + B) = cos A cos B – sin A Sin B

Example 9.14 : (i) Evaluate $\frac{2\sec\phi}{1-\tan^2\phi}$ when $\cos\phi = \frac{\sqrt{3}}{2}$

(ii) Evaluate : $\frac{3\cos\phi - 4\sin\phi}{2\cos\phi + \sin\phi}$ when 3 cot ϕ = 2

where $0^o < \phi < 90^o$

Solution : (i) Given : $\cos\phi = \frac{\sqrt{3}}{2} \Rightarrow \phi = 30^o$

$$\frac{2\sec\phi}{1-\tan^2\phi} = \frac{2\sec(30^o)}{1-\tan^2 30}$$

$$= \frac{2\left(\frac{2}{\sqrt{3}}\right)}{1-\left(\frac{1}{\sqrt{3}}\right)^2} = \frac{\frac{4}{\sqrt{3}}}{\frac{2}{3}} = 2\sqrt{3}$$

(ii) By the given data 3 cot ϕ = 2 $\Rightarrow$ cot ϕ = $\frac{2}{3}$

$$\frac{3\cos\phi - 4\sin\phi}{2\cos\phi + \sin\phi} = \frac{3\frac{\cos\phi}{\sin\phi} - 4}{2\frac{\cos\phi}{\sin\phi} + 1} \quad \text{[Divide the Nr. and Dr. by } \sin\phi]$$

$$= \frac{3\cot\phi - 4}{2\cot\phi + 1} = \frac{3\left(\frac{2}{3}\right) - 4}{2\left(\frac{2}{3}\right) + 1} = \frac{-6}{7}$$

Example 9.15 : Using the formula sin (A – B) = sin A cos B – cos A sin B, find the value of sin 15^o.

Solution : Let A = 45^o, B = 30^o then A – B = 15^o

$$\sin(45^o - 30^o) = \sin 45^o \cos 30^o - \cos 45^o \sin 30^o$$

$$\sin 15^o = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3} - 1}{2\sqrt{2}} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

Example 9.16 : Using the formula tan 2θ = $\frac{2\tan\theta}{1 - \tan^2\theta}$, find the value of tan$60^o$, given that tan 30^o = $\frac{1}{\sqrt{3}}$.

Solution : Putting θ = 30^o in tan 2θ = $\frac{2\tan\theta}{1 - \tan^2\theta}$ we get

$$\tan 2(30^o) = \frac{2\tan 30^o}{1 - \tan^2(30^o)}$$

$$\tan(60^o) = \frac{2\left[\frac{1}{\sqrt{3}}\right]}{1 - \left(\frac{1}{\sqrt{3}}\right)^2}$$

$$\tan(60^o) = \frac{\frac{2}{\sqrt{3}}}{\frac{2}{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

Example 9.17 : Using the formula $\cos 2\theta = 2\cos^2\theta - 1$ find the value of $\cos 60^o$, given that $\cos 30^o = \frac{\sqrt{3}}{2}$.

Solution : Putting $\theta = 30^o$ in $\cos 2\theta = 2\cos^2\theta - 1$, we get

$$\cos 2(30^o) = 2\cos^2 30^o - 1$$

$$\Rightarrow \cos(60^o) = 2\cos^2 30^o - 1$$

$$\Rightarrow \cos 60^o = 2\left(\frac{\sqrt{3}}{2}\right)^2 - 1$$

$$= \frac{3}{2} - 1$$

$$\therefore \cos 60^o = \frac{1}{2}$$

Example 9.18 : Using the formula $\cos 2\theta = 1 - 2\sin^2\theta$, find the value of $\cos 60^o$ being given that $\sin 30^o = \frac{1}{2}$.

Solution : Putting $\theta = 30^o$ in $\cos 2\theta = 1 - 2\sin^2\theta$

$$\Rightarrow \cos 60^o = 1 - 2\sin^2 30^o$$

$$\Rightarrow \cos 60^o = 1 - 2\left(\frac{1}{2}\right)^2$$

$$= 1 - 2\left(\frac{1}{4}\right)$$

$$= 1 - \frac{1}{2}$$

$$\therefore \cos 60^o = \frac{1}{2}$$

Example 9.19: If $\tan\alpha = \frac{1}{2}$ and $\tan\beta = \frac{1}{3}$ by using the formula $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$ prove that $\alpha + \beta = 45^o$.

Solution : Putting $\tan\alpha = \frac{1}{2}$, $\tan\beta = \frac{1}{3}$ in $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$$\tan(\alpha+\beta) = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \left(\frac{1}{2}\right)\times\left(\frac{1}{3}\right)}$$

$$= \frac{\frac{3+2}{6}}{\frac{6-1}{6}} = \frac{\left(\frac{5}{6}\right)}{\left(\frac{5}{6}\right)} = 1$$

$\therefore \tan(\alpha+\beta) = 1$ [since $\tan 45^o = 1$]

$\therefore \alpha + \beta = 45^o$

Example 9.20 : Using the formula $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$ find the value of $\sin 60^o$, given that $\tan 30^o = \frac{1}{\sqrt{3}}$.

Solution : Putting, $\theta = 30^o$ in $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$

we get $\sin 2(30^o) = \frac{2\tan(30^o)}{1+\tan^2(30^o)}$

$$\Rightarrow \quad \sin 60^o = \frac{2\left(\frac{1}{\sqrt{3}}\right)}{1+\left(\frac{1}{\sqrt{3}}\right)^2} = \frac{\frac{2}{\sqrt{3}}}{1+\frac{1}{3}}$$

$$= \frac{\frac{2}{\sqrt{3}}}{\frac{4}{3}} = \frac{\sqrt{3}}{2}$$

$$\therefore \sin 60^o = \frac{\sqrt{3}}{2}$$

Example 9.21 : When $\cos\theta > 0$ solve the following equations.

(i) $2\cos^2\theta = \frac{1}{2}$ (ii) $\frac{\cos\theta}{1-\sin\theta} + \frac{\cos\theta}{1+\sin\theta} = 4$

Solution :

(i) $2\cos^2\theta = \frac{1}{2}$

$\Rightarrow \cos^2\theta = \frac{1}{4}$

$\Rightarrow \cos\theta = \frac{1}{2}$ since $\cos\theta > 0$

$\Rightarrow \theta = 60^o$

(ii) $\frac{\cos\theta}{1-\sin\theta} + \frac{\cos\theta}{1+\sin\theta} = 4$

$\Rightarrow \frac{\cos\theta\,(1+\sin\theta) + \cos\theta\,(1-\sin\theta)}{1-\sin^2\theta} = 4$

$\Rightarrow \frac{\cos\theta + \cos\theta\sin\theta + \cos\theta - \cos\theta\sin\theta}{\cos^2\theta} = 4$

$\Rightarrow \frac{2\cos\theta}{\cos^2\theta} = 4 \Rightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \theta = 60^o$ [since $\cos 60^o = \frac{1}{2}$]

EXERCISE 9.2

1. Find the value of each of the following expressions.
   (i) $4\cot 45^o - \sec^2 60^o + \sin^2 60^o + \cos^2 90^o$
   (ii) $\sin 60^o \cos 30^o - \cos 60^o \sin 30^o + 3\cos 0^o + 5\sin 90^o$
   (iii) $\frac{\tan^2 60^o + 4\sin^2 45^o + \sec^2 30^o + 5\cos^2 90^o}{\operatorname{cosec}(30^o) + \sec(60^o) - \cot^2 30^o}$

2. Verify that

(i) $\sin 90^o = \sin 60^o \cos 30^o + \cos 60^o \sin 30^o$

(ii) $\cos 0^o = \cos^2 45^o + \sin^2 45^o$

3. If $\phi = 30^o$ verify that

(i) $\sin 3\phi = 3 \sin \phi - 4 \sin^3 \phi$

(ii) $\cos 3\phi = 4 \cos^3 \phi - 3 \cos \phi$

4. (i) Using the formula $\cos (A + B) = \cos A \cos B - \sin A \sin B$, evaluate $\cos 75^o$.

(ii) Using the formula $\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$, show that $\sin (75^o) = \frac{\sqrt{3}+1}{2\sqrt{2}}$

(iii) Using the formula $\tan (\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan\alpha \tan \beta}$ find the value of $\tan 15^o$.

5. Solve the following equation for $0^o < \theta \leq 90^o$, $\frac{\cos^2 \theta - 3\cos\theta + 2}{\sin^2 \theta} = 2$

6. When $0^o \leq \theta \leq 90^o$, solve the following :

(i) $2 \cos 3\theta = 1$, (ii) $2 \cos^2\theta + \sin \theta - 2 = 0$

7. (i) If $4 \tan \theta = 5$, evaluate $\frac{2\cos\theta - 3\sin\theta}{3\cos\theta + 2\sin\theta}$

(ii) If $\cot \theta = \frac{a}{b}$, find the value of $\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$.

8. If $\operatorname{cosec} \phi = \sqrt{2}$, show that $\frac{2\sin^2 \phi + 3\cot^2 \phi}{4 (\tan^2 \phi - \cos^2 \phi)} = 2$.

9. Find x from the following equation : $\tan^2 45^o - \cos^2 60^o = x . \sin 45^o \cos 45^o \cot 30^o$

9.3 TRIGONOMETRIC TABLES

In the previous chapter, we have mainly used the trigonometric ratios of some standard angles viz. $30^o, 45^o, 60^o, 90^o$.

For applying the results of trigonometry to different practical situations, we need trigonometric ratios of different angles other than the standard angles. The values of different trigonometric ratios have been computed for different angles from 0^o to 90^o correct to three or four places of decimals and are presented in tabular form. These tables are known as the Trigonometric tables.

We know that $1^o = 60'$

$$\Rightarrow \quad 1' = \left(\frac{1}{60}\right)^o$$

$$\Rightarrow \quad 6' = \left(\frac{6}{60}\right)^o$$

$$\Rightarrow \quad 6' = (0.1)^o$$

$$\Rightarrow \quad (0.1)^o = 6'$$

Degrees may be further divided into minutes and seconds, but that division is not as universal as it used to be. Parts of a degree are now frequently referred to decimals. For instance seven and a half degree is now usually written as 7.5^o. Each degree is divided into 60 equal parts called minutes. So seven and a half degree can be called as 7 degrees and 30 minutes, written as $7^o\ 30'$

$$\therefore (7.5)^o = 7^o\ 30' \qquad [\text{since } (0.1)^o = 6']$$

Note :

1. As the measure of angle increases, the sine value increases till 90^o

Example : $45^o > 30^o \Rightarrow \sin 45^o > \sin 30^o \Rightarrow \frac{1}{\sqrt{2}} > \frac{1}{2}$

2. As the measure of angle increases, the cosine value decreases till 90^o

Example : $45^o > 30 \Rightarrow \cos 45^o < \cos 30^o \Rightarrow \frac{1}{\sqrt{2}} < \frac{\sqrt{3}}{2}$

3. As the measure of angle increases the tangent value increases to infinity till 90^o.

Example : $45^o > 30^o \Rightarrow \tan 45^o > \tan 30^o \Rightarrow 1 > \frac{1}{\sqrt{3}}$

4. As the measure of angle increases, the cosecant value decreases till 90^o

Example : $45^o > 30^o \Rightarrow \text{cosec } 45^o < \text{cosec } 30^o \Rightarrow \sqrt{2} < 2$

5. As the measure of angle increases the secant value increases till 90^o

Example : $45^o > 30 \Rightarrow \sec 45^o > \sec 30^o \Rightarrow \sqrt{2} > \frac{2}{\sqrt{3}}$

6. As the measure of angle increases, the cotangent value decreases till 90^o

Example : $45^o > 30^o \Rightarrow \cot 45^o < \cot 30^o \Rightarrow 1 < \sqrt{3}$

9.3.1 Trigonometric Tables

Trigonometric tables are those which provide the values of sines, cosines and tangents of all the acute angles approximately. The learners shall refer Mathematical tables for Natural sines, natural cosines and natural tangents. These tables give the values, correct upto three or four places of decimals of the three trigonometric ratios for the angles from 0^o to 90^o spaced at equal intervals of $6' = (0.1)^o$

Each trigonometric table is divided into 12 columns. The first column from the left contains degrees from 0^o to 89^o. Then there are ten columns headed by 0', 6', 12', 18', 24', 30', 36', 42', 48' and 54' respectively. The twelfth column is the column containing the mean differences and is subdivided into five columns headed by 1', 2', 3', 4' and 5' respectively.

The working rule to find the value of the required trigonometric ratio of a given angle from the table is as follows.

Step (1) : Select the angle for which the trigonometric ratio has to be found.

Step (2) : Check whether the number of minutes in the given angle is a multiple of 6. If not, convert it to the nearest multiple of 6. [e.g., 26' = 24' + 2']

Step (3) : Let the angle be $x^o (6p + q)'$
where $p = 0, 1, 2, 3, \ldots, 9$ & $q = 0, 1, 2, 3, 4, 5$.

Step (4) : Select the trigonometric table required for the given ratio.

Step (5) : Move vertically down the first column of the table to reach x^o and then horizontally to reach the column headed by $(6p)'$. Get the number at intersection of rows and columns.

Step (6) : If $q = 0$, then the number obtained above is the value of the required trigonometric ratio.

Else, proceed further.

Step (7) : Move further horizontally to reach the mean difference column headed by *q*'. Get the number at the intersection of row and column.

Step (8) : (i) If the trigonometric ratio is sine (or) tangent then **add** the mean difference obtained above with the value obtained in step (5).

(ii) If the trigonometric ratio is cosine then **subtract** the mean difference obtained above from the value obtained in step (5).

Example 9.23 : Find the value of (i) sin (3^o) (ii) sin (5^o 18') (iii) sin (7^o 50')

Solution :

Natural sines Table.

Degree	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	Mean difference				
	$(0.0)^o$	$(0.1)^o$	$(0.2)^o$	$(0.3)^o$	$(0.4)^o$	$(0.5)^o$	$(0.6)^o$	$(0.7)^o$	$(0.8)^o$	$(0.9)^o$	1	2	3	4	5
3 →	0.0523														
4															
5 →				0.0924											
6															
7 →									0.1357			6			

(i) Look in the first column of the trigonometric table of natural sines for 3^o.

The number at the intersection of the row containing 3^o with 0' column (because there are 0 minutes) is 0.0523.

Hence sin (3^o) = 0.0523.

(ii) sin (5^o 18')

Look in the first column of the trigonometric table of natural sines for 5^o.

Then move horizontally to arrive at a column headed by 18'. The number at the intersection of the row containing 5^o and the column headed by 18' is 0.0924

Hence, sin (5^o 18') = 0.0924.

(iii) sin (7^o 50')

We have 7^o 50' = 7^o 48' + 2'

Look in the first column of the trigonometric table of natural sines for 7^o.

Then move horizontally to arrive at a column headed by 48'. The number at the intersection of these two is 0.1357.

Further move horizontally to arrive at a column headed by 2' in the mean difference column. The number at the intersection is 6 [(i.e.,) 0.0006] This number is added to 0.1357 to get the required value as shown below

$$\sin(7^o\ 48') = 0.1357$$

$$\text{Mean Diff. for } 2' = 0.0006$$

$$\therefore \sin(7^o\ 50') = 0.1363$$

Example 9.24 : Using Trigonometric tables, write the values of each of the following.

(i) cos (37°) (ii) cos (39° 30') (iii) cos (41° 35')

Solution :

Natural Cosines

Degree	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	Mean difference				
	(0.0)°	(0.1)°	(0.2)°	(0.3)°	(0.4)°	(0.5)°	(0.6)°	(0.7)°	(0.8)°	(0.9)°	1	2	3	4	5
37 →	0.7986														
38															
39 →						0.7716									
40															
41 →						0.7490									10

(i) cos (37°)

Look in the first column of the trigonometric table of natural cosines for 37°

The number at the intersection of the row containing 37° with 0' column (because there are 0 minutes) is 0.7986

Hence cos (37°) = 0.7986.

(ii) cos (39° 30')

Look in the first column of the trigonometric table of natural cosines for 39°

Then move horizontally to arrive at a column headed by 30'. The number at the intersection of these two is 0.7716.

Hence cos (39° 30') = 0.7716

(iii) cos (41° 35')

We have 41° 35' = 41° 30' + 5'

Look in the first column of the trigonometric table of natural cosines for 41°.

Then move horizontally to arrive at a column headed by 30'. The number at the intersection of these two is 0.7490

Further move horizontally to arrive at a column headed by 5' in the mean difference column.

The number at the intersection is 10 [(i.e.,) 0.0010]

This number is subtracted from 0.7490 to get the required value

cos (41° 30') = 0.7490 –

Mean difference for 5' = 0.0010

∴ cos (41° 35') = 0.7480

Example 9.25 : Using Trigonometric tables, write the value of tan (28°26')

Solution : tan (28° 26')

We have 28° 26' = 28° 24' + 2'

From the table of natural tangent,

tan (28° 24') = 0.5407 +

Mean diff. for 2' = 0.0008

tan (28° 26') = 0.5415

Example 9.26 : Using Trigonometric tables, write the value of cot (20° 19')

Solution :

We know that cot (20° 19') = tan (90° – 20° 19')

= tan (69° 41')

From the table of natural tangents

tan 69° 36' = 2.6889

Mean difference for 5' = 0.0119

∴ tan (69° 41') = 2.7008

Example 9.27 : Using Trigonometric tables, find the value of each of the following.

(i) sec (50° 26') (ii) cosec (25° 15')

Solution : (i) $\sec (50^\circ\ 26') = \dfrac{1}{\cos(50^\circ\ 26')}$

$$\cos(50^\circ 24') = 0.6374 -$$

$$\text{Mean difference for } 2' = \underline{0.0004}$$

$$0.6370$$

$$\therefore \sec(50^\circ 26') = \frac{1}{0.6370} = 1.56985 = 1.5699$$

(ii) $$\operatorname{cosec}(25^\circ 15') = \frac{1}{\sin(25^\circ 15')}$$

$$\sin(25^\circ 12') = 0.4258 +$$

$$\text{Mean difference for } 3' = \underline{0.0008}$$

$$0.4266$$

$$\therefore \operatorname{cosec}(25^\circ 15') = \frac{1}{0.4266} = 2.3441$$

9.3.2 The angle for a given trigonometric ratio

If the value of the trigonometric ratio is given, then we can easily find the corresponding angle with the help of the trigonometric table.

Example 9.28 : Find the angle θ in each of the following cases :

(i) $\sin\theta = 0.8988$ (ii) $\tan\theta = 0.5058$ (iii) $\cos\theta = 0.7951$

Solution : (i) We have $\sin\theta = 0.8988$

From the table of natural sines, we find that the number 0.8988 is corresponding to $\sin 64^\circ$

Hence the required angle $\theta = 64^\circ$

(ii) We have $\tan\theta = 0.5058$

From the table of natural tangents we find that the number nearest to 0.5058 is 0.5051 which is corresponding to $\tan(26^\circ 48')$ (i.e.,) $\tan(26^\circ 48') = 0.5051$

The difference between the given value 0.5058 and the value 0.5051 is 0.0007.

From the mean difference column of the row headed by 26°, we find that 0.0007 corresponds to an increase of 2' in the angle.

Hence the required angle $= 26^\circ 48' + 2' = 26^\circ 50'$

(iii) We have $\cos\theta = 0.7951$

From the table of natural cosines we find that the number nearest to 0.7951 is 0.7955 which is corresponding to $\cos(37^\circ 18')$

(i.e.,) cos (37° 18') = 0.7955

The difference between the given value 0.7951 and the value 0.7955 is 0.0004

From the mean difference column of the row headed by 37°, we find that 0.0004 corresponds to an increase 2' in the angle

Hence the required angle = 37° 18' +2' = 37° 20'

9.3.3 Solution of a right triangle

A triangle consists of six parts viz. three sides and three angles. These six parts are known as the elements of the triangle. From the plane geometry, we know that if three of the six elements of a triangle are known, (atleast one of which must be a side), then other three elements can be determined. The process of finding the remaining three elements of a triangle when its other three elements are given is called the solution of the triangle.

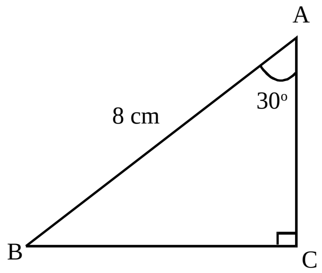
Regular Polygon : A polygon is a regular polygon if its all sides are equal in length and all its interior angles are equal.

Note : 1. Each interior angle of a regular polygon of '*n*' sides is equal to $\left(\frac{n-2}{n}\times 180\right)^{o}$

2. If an n sided regular polygon is inscribed in a circle, then the each central angle subtended by each of its side is equal to $\left(\frac{360}{n}\right)^{o}$

Example 9.29 : Solve the right triangle ABC in which $\angle C = 90^o$, $\angle A = 30^o$ and AB = 8 cm.

Solution : We know that, In Δ ABC



$$\angle A + \angle B + \angle C = 180^o$$

$$30^o + \angle B + 90 = 180^o$$

$$\angle B = 180^o - 120^o$$

$$\angle B = 60^o$$

In Δ ABC, $\sin 30^o = \frac{BC}{AB}$

$$\frac{1}{2} = \frac{BC}{8} \Rightarrow BC = 4 \text{ cm}$$

and $\cos 30^o = \frac{AC}{AB}$

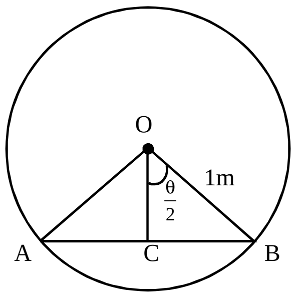
$$\frac{\sqrt{3}}{2} = \frac{AC}{8}$$

Hence B = 60^o, BC = 4 cm, AC = $4\sqrt{3}$ cm.

Example 9.30 : Find the length of a side of a regular polygon inscribed in a circle of radius 1 m if it has 24 sides. Give the answer in cm

Solution : We know that

If the polygon has n sides, then the angle subtended by each side at the centre of the circle is



$$\theta = \left(\frac{360}{n}\right)$$

Here $n = 24 \Rightarrow \theta = \left(\frac{360}{24}\right)^o \Rightarrow \theta = 15^o$

In the figure , OC ⊥ AB.

In Δ OCB [right angled triangle]

$$\sin\left(\frac{\theta}{2}\right) = \frac{BC}{OB}$$

$$\sin\left(\frac{15}{2}\right) = \frac{BC}{1} \qquad \text{[since OB = 1 m.]}$$

$$\sin (7.5)^o = BC$$

$$\Rightarrow BC = \sin (7^o\ 30') \qquad [\text{since } (0.5)^o = 30']$$

$$BC = 0.1305$$

$$\text{Now } AB = 2\ BC = 2\ (0.1305)$$

$$AB = 0.2610 \text{ m}$$

$$AB = 26.1 \text{ cm}$$

∴ The side of a regular polygon inscribed in a circle is 26.1 cm.

Example 9.31 : Find the area of an isosceles triangle with base 10 cm and vertical angle 67^o.

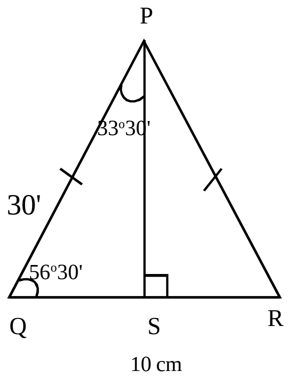
Solution : Let PQR be the given isosceles triangle in which PQ = PR and Base QR = 10 cm.

Draw PS $\perp$ QR $\Rightarrow$ QS = SR = 5cm

[since PS is the median to QR & PS is the angular bisector of $\angle$QPR]

$$\angle QPS = \frac{1}{2}\angle QPR = \frac{67^o}{2} = (33.5)^o$$

$$= 33^o\ 30'$$



In Δ PQS We have

$\angle PQS + \angle QSP + \angle SPQ = 180^o$

$\angle PQS + 90^o + 33^o\ 30' = 180^o$

$\angle PQS = 180^o - 123^o\ 30' = 56^o\ 30'$

In right angled Δ PSQ.

$$\tan (56^o\ 30') = \frac{PS}{QS}$$

$$1.5108 = \frac{PS}{5}$$

$$1.5108 \times 5 = PS \Rightarrow PS = 7.5540$$

Now area of Δ PQR = $\frac{1}{2} \times QR \times PS = \frac{1}{2} \times 10 \times 7.5540 = \frac{75.54}{2} = 37.77\ cm^2$

EXERCISE 9.3

Using trigonometric tables, find the values of the following :

1.	(i) sin 32^o	(ii) sin 50^o	(iii) sin 43^o 18'
	(iv) sin 72^o 32'	(v) sin $(12.5)^o$	(vi) sin $(43.1)^o$
2.	(i) cos 45^o	(ii) cos (48^o 18')	(iii) cos (21^o 24')
	(iv) cos$(48.7)^o$	(v) cos (88^o 44')	(vi) cos (26^o 55')
3.	(i) tan 50^o	(ii) tan (17^o 30')	(iii) tan (49^o 12')
	(iv) tan $(15.2)^o$	(v) tan (10^o 34')	(vi) tan (71^o 20')

4. (i) cosec 65^o (ii) cosec $(77^o\ 15')$ (iii) cosec $(20^o\ 19')$
(iv) cosec $(75^o\ 48')$ (v) cosec $(30.8)^o$ (vi) cosec $(51.4)^o$

5. (i) sec 40^o (ii) sec $(40^o\ 36')$ (iii) sec $(68^o\ 10')$
(iv) sec $(72^o\ 15')$ (v) Sec $(15.3)^o$ (vi)sec $(21.7)^o$

6. (i) cot 12^o (ii) cot $(45^o\ 36')$ (iii) cot $(42^o\ 35')$
(iv) cot $(21.3)^o$ (v) cot $(70.5)^o$ (vi) cot $(15^o\ 18')$

7. (i) $\tan(51^o\ 15') + \cot(25^o\ 18')$ (ii) $\dfrac{\sin 40^o + \cos 20^o}{\tan(30^o 15')}$

8. Solve the triangle ABC in which $\angle A = 25^o\ 30'$, $\angle B = 90^o$ and AB = 10 cm.

9. Find the area of a right angled triangle with hypotenuse 8 cm and one of the acute angle 57^o.

10. Find the area of an isosceles triangle with base 10 cm and vertical angle 47^o.

11. Find the length of the chord of a circle of radius 6cm, subtending at the centre an angle of 144^o.

12. Find the radius of the incircle of a regular polygon of 18 sides each of length 60cm.

13. Compare the lengths of chords of a circle with radius 3cm subtending angles 108^o, 72^o at the centre of the circle.

9.4 HEIGHTS AND DISTANCES

One important practical application of trigonometry is to find the heights of objects and the distances between them without actual measurement.

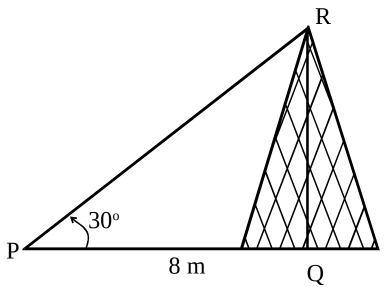
Many times we are required to find the distance between two objects and to measure the height of a mountain, building, tower, width of a gulf etc.It is practically impossible but we can determine those by using trigonometric ratios. Here trigonometry helps us in finding out the heights of objects or distances between inaccessible regions.

The problem of finding heights and distances is solved by observation of angles subtended by those objects at the eye of the observer.

In section 9.2 we have learnt that in a right triangle if one side and another part (side or angle) are known, we can find the remaining parts of the triangle.

Suppose we wish to determine the height of a tower without actually measuring it. We could stand on the ground at a point P at some distance say 8m from the foot Q of the tower. Suppose the measure of $\angle QPR = 30^o$. Then we can find the height QR of the tower by using trigonometric method

In right angled Δ PQR.



$$\tan 30^{o} = \frac{QR}{PQ}$$

$$\frac{1}{\sqrt{3}} = \frac{QR}{8} \Rightarrow QR = \frac{8}{\sqrt{3}} \text{ m}$$

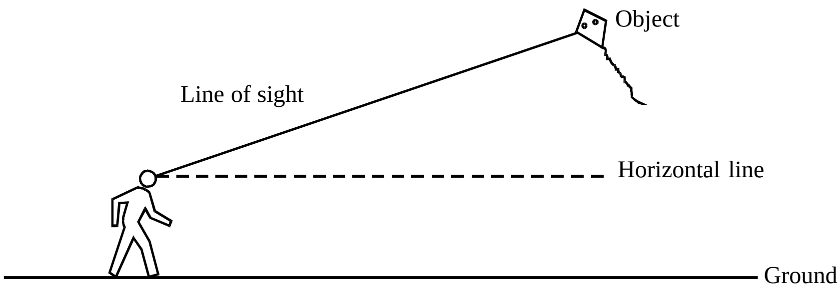
$\therefore$ The height of the Tower is $\frac{8}{\sqrt{3}}$ m.

Thus we are able to find the height of the tower using trigonometric ratios.

Before we proceed to solve problems of the above type, let us first define a few terms.

9.4.1 Line of sight

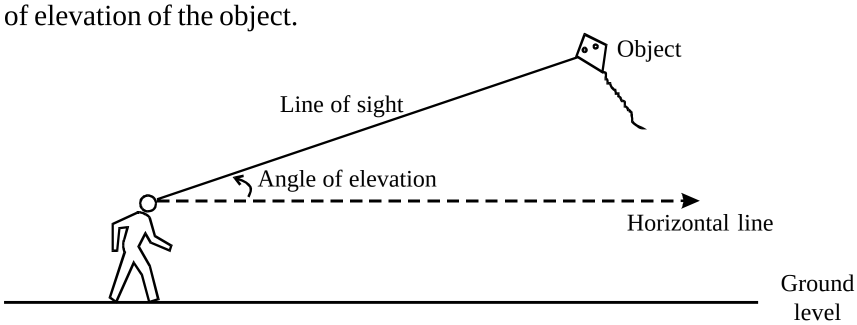
Suppose we are viewing an object standing on the ground. Clearly the line of sight (or line of vision) to the object is the line from our eyes to the object.



9.4.2 Angle of Elevation

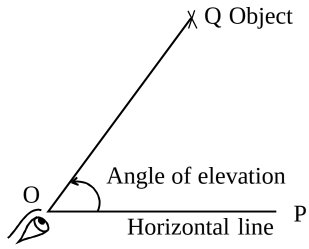
If the object is above the horizontal level of the eyes (i.e., if it is above the eye - level) We have to turn our head upwards to view the object.

In this process, our eyes move through an angle. Such an angle is called the angle of elevation of the object.



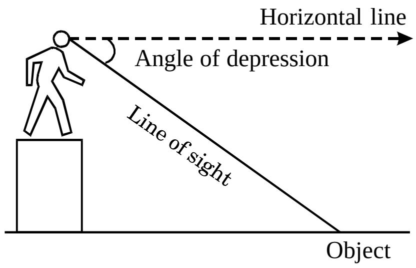
Let O be an eye of an observer and OP be the horizontal line drawn through O.

Let Q be an object then Q is above OP as in figure. ∠POQ is called the angle of elevation of Q.



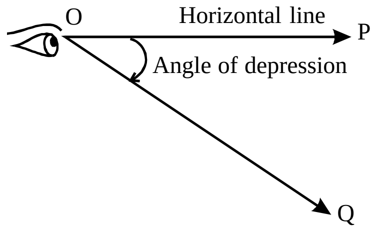
9.4.3 Angle of depression

Suppose a boy standing on the roof of a building, observes a ball lying on the ground at some distance from the building. In this case he has to move his head downwards to view the ball.

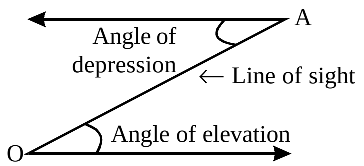


In this process our eyes move through an angle. Such an angle is called the angle of depression of the object.

Let O be an eye of an observer and OP be the horizontal line drawn through O. Let Q be an object. Here ∠POQ is called the angle of depression of Q.



Note: Obviously, the angle of elevation of a point A as seen from a point O is equal to the angle of depression of O as seen from A.



Example 9.32 : From a cliff 150 m above the shore line the angle of depression of a ship is $19^o\ 30'$. Find the distance from the ship to a point on the shore directly below the observer.

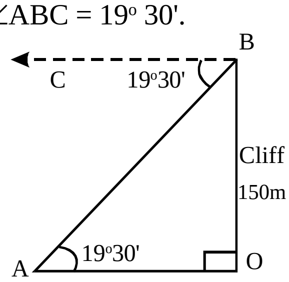
Solution : Let OB be the cliff of height 150 m and A be the position of the ship. The angle of depression of the ship is $19^o\ 30'$ Then $\angle ABC = 19^o\ 30'$.

$$\Rightarrow \angle OAB = 19^o 30'$$

In right angled Δ AOB, $\cot(19^o\ 30') = \frac{OA}{OB} = \frac{OA}{150}$

$$\therefore\ OA = 150 \cot(19^o\ 30')$$

$$= 150 \tan(90^o - 19^o\ 30')$$



$= 150 \tan (70^o 30')$

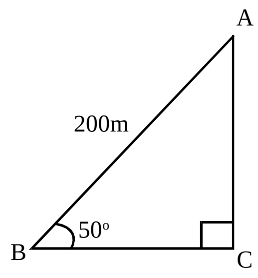
$= 150 (2.8239) = 423.59$ m.

Example 9.33 : A kite is flying with a thread 200 m long. If the thread makes an angle of 50^o with the horizontal, find the height of the kite above the ground, assuming that there is no slack in the thread.

Solution : Let A be the position of the kite at a height AC. Here AB = 200 m

$\angle CBA = 50^o$

In right angled triangle BCA.



$$\sin 50^o = \frac{AC}{AB}$$

$$0.7660 = \frac{AC}{200}$$

$$\Rightarrow AC = 200 \times 0.7660$$

$$AC = 153.2$$

$$\therefore \text{The height of the kite} = 153.2 \text{ m.}$$

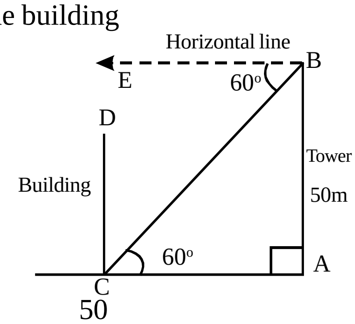
Example 9.34 : The measure of angle of depression of the bottom of a building on a level ground from the top of a tower 50 m high is 60^o. How far is the building from the tower ?

Solution : Let AB be the tower and CD be the building

Here AB = 50 m.

The angle of depression $\angle CBE = 60^o$

$\Rightarrow \angle BCA = 60^o$



In right angled Δ CAB, $\tan 60^o = \frac{AB}{AC}$

$$\sqrt{3} = \frac{50}{AC} \Rightarrow AC = \frac{50}{\sqrt{3}}$$

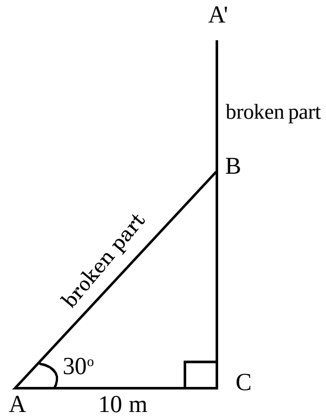
$$AC = \frac{50 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{50 \times 1.732}{3}$$

$$= 28.9 \text{ m (approx)}$$

Example 9.35 : A pole being broken by the wind, the top struck the ground making an angle of 30° and at a distance of 10m from the foot of the pole. Find the whole height of the pole.

Solution : Let A'BC be the pole.

When broken at B by the wind, let its top A' strikes the ground so that $\angle CAB = 30^o$ and AC = 10 m



In right angled Δ ACB, $\tan 30^o = \frac{BC}{AC}$

$$\frac{1}{\sqrt{3}} = \frac{BC}{10}$$

$$\Rightarrow \quad BC = \frac{10}{\sqrt{3}} \text{ m}$$

Also $\cos 30^o = \frac{AC}{AB}$

$$\frac{\sqrt{3}}{2} = \frac{10}{AB}$$

$$\Rightarrow \quad AB = \frac{20}{\sqrt{3}} \text{ m}$$

The height of the pole = AB + BC

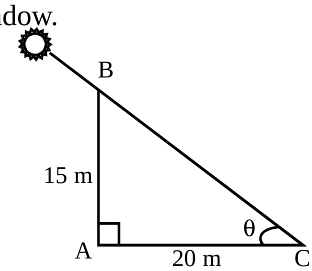
$$= \frac{20}{\sqrt{3}} + \frac{10}{\sqrt{3}} = 10\sqrt{3}$$

= 17.32 cm (approx)

Example 9.36 : A vertical pole is 15m high and the length of its shadow is 20 m. What is the angle of elevation of the sun.

Solution : Let AB be the pole and AC be the shadow.

Here AB = 15m. ; AC = 20 m.



In right angled Δ BAC, let $\angle ACB = \theta$

$$\therefore \tan\theta = \frac{AB}{AC} = \frac{15}{20}$$

$$\tan\theta = 0.75$$

$\Rightarrow \theta = 36^o\ 53'$ (from the trigonometric table).

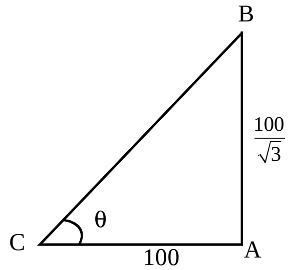
Example 9.37 : A tower is $\frac{100}{\sqrt{3}}$ metres high. Find the angle of elevation if the point of observation is 100 metres away from its foot.

Solution : Let AB be the tower of height $\frac{100}{\sqrt{3}}$ metres and C be a point at a distance of 100 metres from foot of the tower.

Let θ be the angle of elevation of the top of the tower from point C.

In rt. Δ CAB, $\tan\theta = \frac{AB}{AC} = \frac{\frac{100}{\sqrt{3}}}{100} = \frac{1}{\sqrt{3}}$

$$\therefore \theta = 30^o = \frac{\pi}{6}$$



Hence the angle of elevation of the top of the tower from a point 100 metres away from its foot is 30^o.

Example 9.38 **:** A person standing on the bank of a river observes that the angle of elevation of the top of a tree standing on the opposite bank is 60^o. When he was 40m away from the bank he finds that the angle of elevation to be 30^o. Find (i) The height of the tree (ii) The width of the river, correct to two decimal places.

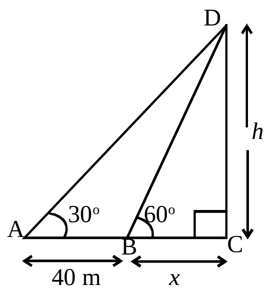
Solution : Let CD = h be the height of the tree and BC = x be the breadth of the river. From the figure ∠DAC = 30^o and ∠DBC = 60^o

In right angled Δ BCD, $\tan 60^o = \frac{DC}{BC} \Rightarrow \sqrt{3} = \frac{h}{x} \Rightarrow h = \sqrt{3}\,x$ ... (1)

From the right angled Δ ACD,

$$\tan 30^o = \frac{DC}{AC}$$

$$\frac{1}{\sqrt{3}} = \frac{h}{40+x} \Rightarrow \sqrt{3}\,h = 40 + x \text{ ... (2)}$$



Substituting the value of h in (2) we get

$$\sqrt{3}\,(x\sqrt{3}) = 40 + x$$

$$\Rightarrow \quad 3x = 40 + x$$

$$\Rightarrow \quad x = 20$$

from (1) we get $h = \sqrt{3}\,(20) = 20\,(1.732) = 34.64$.

$\therefore$ height of the tree = 34.64 m.

width of the river = 20 m.

Example 9.39 : As observed from the top of a light house, 100 m high above sea level, the angle of depression of a ship, sailing directly towards it, changes from 30^o to 45^o. Determine the distance travelled by the ship during the period of observation.

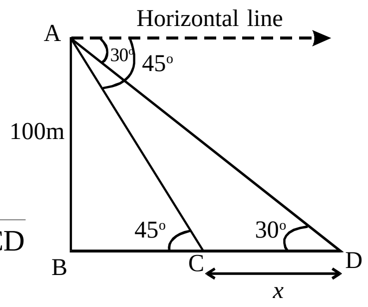
Solution : Let AB = 100 m be the height of the light house.

Let C and D be the two positions of the ship.

Let CD = x be the distance travelled by the ship during the period of observation.

In right angled Δ ABC, $\quad \tan 45^o = \dfrac{AB}{BC}$

$$1 = \frac{AB}{BC} \Rightarrow BC = AB \Rightarrow BC = 100$$



In right angled Δ ABD, $\quad \tan 30^o = \dfrac{AB}{BD}$

$$\frac{1}{\sqrt{3}} = \frac{AB}{BC + CD}$$

$$\Rightarrow \quad BC + CD = AB\sqrt{3}$$

$$\Rightarrow \quad 100 + x = 100\sqrt{3}$$

$$\Rightarrow \quad x = 100\sqrt{3} - 100$$

$$= 100\,(\sqrt{3} - 1)$$

$$= 100\,(0.732)$$

$$x = 73.2$$

$\therefore$ The distance travelled by the ship from D to C is 73.2 m (approx)

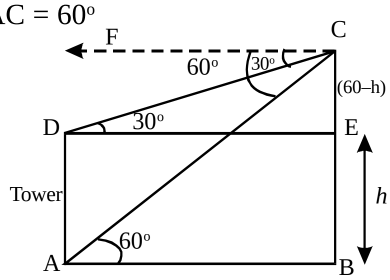
Example 9.40 : From the top of a building 60 m high the angle of depression of the top and the bottom of a tower are observed to be 30^o and 60^o. Find the height of the tower.

Solution : Let AD = h be the height of the tower.

BC be the height of the building. Here BC = 60 m

From the figure $\angle DCF = 30^o \Rightarrow \angle EDC = 30^o$

and $\angle ACF = 60^o \Rightarrow \angle BAC = 60^o$



In Δ DEC, $\tan 30^o = \frac{EC}{DE}$

$$\frac{1}{\sqrt{3}} = \frac{60-h}{DE}$$

$$\Rightarrow DE = (60-h)\sqrt{3}$$

In right angled Δ ABC, $\tan 60^o = \frac{BC}{AB} = \frac{60}{DE}$

$$\Rightarrow \sqrt{3} = \frac{60}{(60-h)\sqrt{3}}$$

$$\Rightarrow 3(60-h) = 60$$

$$\Rightarrow 180-3h = 60$$

$$\Rightarrow 120 = 3h \Rightarrow h = 40$$

$\therefore$ height of the tower is 40 m.

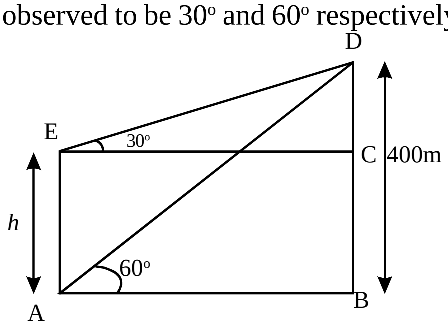
Example 9.41 : From the top and bottom of the tower the angle of elevation of the top of a cliff with height 400 m are observed to be 30^o and 60^o respectively. Determine the height of the tower.

Solution : Let AE be the tower,

AE = h

Let BD be the height of the cliff.

Here BD = 400 m.



From the figure, $\angle BAD = 60^o$ and $\angle CED = 30^o$

In right angled Δ ABD

$$\tan 60^o = \frac{BD}{AB}$$

$$\Rightarrow \sqrt{3} = \frac{400}{AB}$$

$$\Rightarrow AB = \frac{400}{\sqrt{3}}$$

In right angled Δ ECD

$$\tan 30^o = \frac{DC}{EC}$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{400-h}{EC}$$

$$= \frac{400-h}{AB}$$

$$\frac{1}{\sqrt{3}} = \frac{400-h}{\frac{400}{\sqrt{3}}}$$

$$\Rightarrow 400 = 3\,(400 - h)$$

$$3h = 800$$

$$h = 266.67 \text{ m}$$

∴ Required height of the tower is 266.67 m.

Example 9.42: From a window, 20 m above the ground the angle of elevation of the top of a tower is x^o where $\tan x^o = \frac{5}{2}$ and the angle of depression of the foot of the tower is y^o where $\tan y^o = \frac{1}{4}$. Calculate the height of the tower in metres.

Solution : Let A be the window, CE be the height of the tower.

Here AB = 20 m. $\Rightarrow$ CD = 20 m

$\angle DAE = x^o$ and $\angle DAC = y^o \Rightarrow \angle ACB = y^o$ (alternate interior angles).

In right angled Δ ADE

$$\tan x^o = \frac{DE}{AD}$$

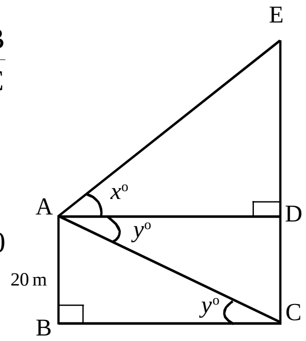
$$\frac{5}{2} = \frac{DE}{AD} \Rightarrow DE = \frac{5}{2}AD$$

In right angled Δ ABC

$$\tan y^o = \frac{AB}{BC}$$

$$\frac{1}{4} = \frac{20}{BC}$$

$$\Rightarrow BC = 80$$



Here $DE = \frac{5}{2}AD$

$= \frac{5}{2}$ BC (since AD = BC)

$= \frac{5}{2}$ (80)

DE = 200

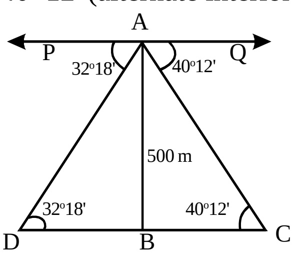
Height of the tower = CD + DE = 20 + 200 = 220 m.

Example 9.43 : An aeroplane at an altitude of 500 m observes the angles of depression of opposite point on the two banks of a river to be 32^o 18' and 40^o 12' respectively. Find in metres, the width of the river.

Solution : Let A be the position of the aeroplane and CD be the width of the river.

From the figure : $\angle DAP = 32^o\ 18' \Rightarrow \angle ADB = 32^o\ 18'$

and $\angle CAQ = 40^o\ 12' \Rightarrow \angle ACB = 40^o\ 12'$ (alternate interior angles).



In right angled ΔDBA

$\cot (32^o\ 18') = \frac{DB}{AB}$

$\Rightarrow$ DB = AB cot $(32^o\ 18')$

$= 500 \tan (90^o - 32^o18')$

$= 500 \tan (57^o\ 42')$

= 500 (1.5818)

DB = 790.9

In right angled Δ ABC

$\cot (40^o\ 12') = \frac{BC}{AB}$

$BC = 500 \tan (90^o - 40^o12')$

$= 500 \tan (49^o\ 48')$

= 500 (1.1833)

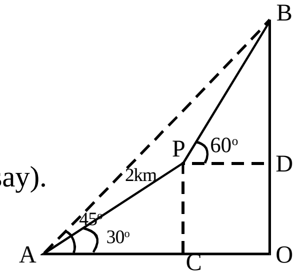
$\therefore$ Width of the river DC = DB + BC

= 790.9 + 591.65

= 1382.55 m.

Example : 9.44 At the foot of a mountain the elevation of its summit is 45^o, after ascending 2 km towards the mountain up a slope of 30^o inclination, the elevation is found to be 60^o. Find the height of the mountain.

Solution : Let A be the foot and B be the summit of the mountain AOB



In right angled Δ AOB, $\angle OAB = 45^o$

$\angle ABO = 45^o$

$\Rightarrow AO = OB = h$ (say).

In right angled Δ ACP

$$\sin 30^o = \frac{PC}{AP}$$

$$\frac{1}{2} = \frac{PC}{2} \Rightarrow PC = 1$$

$$\Rightarrow DO = 1$$

Also $$\cos 30^o = \frac{AC}{AP}$$

$$\frac{\sqrt{3}}{2} = \frac{AC}{2}$$

$$\Rightarrow AC = \sqrt{3}$$

Now h = OB = OA = OC + CA

$$\Rightarrow h = OC + \sqrt{3} \qquad \Rightarrow OC = h - \sqrt{3}$$

$$\Rightarrow PD = h - \sqrt{3}$$

In right angled Δ PDB

$$\tan 60^o = \frac{BD}{PD} \Rightarrow \sqrt{3} = \frac{BD}{h-\sqrt{3}} \Rightarrow \sqrt{3} = \frac{h-1}{h-\sqrt{3}}$$

$$\Rightarrow \sqrt{3}h - 3 = h - 1$$

$$\Rightarrow \sqrt{3}h - h = 2$$

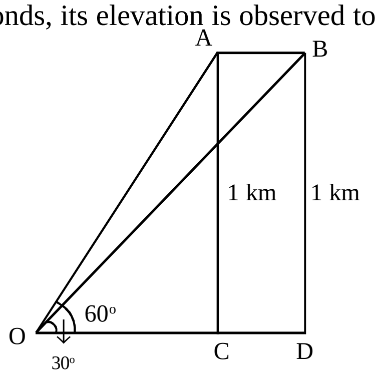
$$h = \frac{2}{\sqrt{3}-1}$$

$$= \frac{2}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$= \sqrt{3} + 1 = 1.732 + 1 = 2.732.$$

Hence the height of the mountain is 2.732 km.

Example 9.45 : An aeroplane flying horizontally 1 km above the ground is observed at an elevation of 60°. After 10 seconds, its elevation is observed to be 30°. Find the speed of the aeroplane in km/hr.



Solution : Let O be the point of observation and A be the position of the aeroplane such that $\angle AOC = 60^o$ and AC = 1 km.

After 10 seconds, let B be the postion of the aeroplane such that $\angle BOD = 30^o$ and BD = 1 km.

In right angled Δ OCA,

$$\tan 60^o = \frac{AC}{OC}$$

$$\sqrt{3} = \frac{1}{OC}$$

$$\Rightarrow \quad OC = \frac{1}{\sqrt{3}}$$

In right angled Δ ODB,

$$\tan 30^o = \frac{BD}{OD}$$

$$\frac{1}{\sqrt{3}} = \frac{1}{OD}$$

$$\Rightarrow \quad OD = \sqrt{3}$$

Now $\quad CD = OD - OC$

$$= \sqrt{3} - \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}$$

Distance covered by the aeroplane in 10 sec = $\frac{2}{\sqrt{3}}$ km.

Time taken = 10 sec. = $\frac{10}{3600}$ hrs = $\frac{1}{360}$ hr.

$$\text{Speed of the aeroplane} = \frac{\text{Distance}}{\text{time}}$$

$$= \frac{\frac{2}{\sqrt{3}}}{\frac{1}{360}} = \frac{2}{\sqrt{3}} \times 360$$

$$= \frac{720}{\sqrt{3}} = \frac{720\sqrt{3}}{\sqrt{3}\times\sqrt{3}}$$

$$= \frac{720\sqrt{3}}{3} = 240 \times 1.732$$

$$= 415.68 \text{ km/hr}$$

EXERCISE 9.4

1. Determine the angle of elevation of the sun when the length of the shadow of a pole is $\sqrt{3}$ times the height of the pole.
2. An observer 1.5 m tall is 28.5m away from a tower 30m high. Determine the angle of elevation of the top of the tower from his eye.
3. A kite is flying at a height of 75 metres from the level ground, attached to a string inclined at 60^o to the horizontal. Find the length of string.
4. A tree 12 m high is broken by the wind in such a way that its top touches the ground and makes an angle $\frac{\pi}{4}$ radians with the ground. At what height from the bottom of the tree is broken by the wind ?
5. The angle of elevation of a ladder leaning against a wall is 60^o and the foot of the ladder is 9.5m away from the wall. Find the length of the ladder.
6. A bridge across a river makes an angle of 45^o with the river bank. If the length of the bridge across the river is 150 m, what is the width of the river ?
7. From the tower 80 metres high, the angle of depression of a car on the ground is found to be 30^o 12'. Find the distance of the car from the foot of the tower.
8. The shadow of a tower, when the angle of elevation of the sun is 45^o is found to be 10 metres longer than when it is 60^o. Find the height of the tower.
9. The angle of elevation of a tower at a point is 45^o. After going 20 metres towards the foot of the tower the angle of elevation of the tower becomes 60^o. Calculate the height of the tower.
10. The angle of elevation of a Jet plane from a point P on the ground is 60^o. After 15 seconds, the angle of elevation changes to 30^o. If the Jet is flying at a speed of 720 km/hr, find the height at which the jet is flying.
11. From the top of a tree, the angle of depression of an object on the horizontal ground is found to be 60^o. On descending 20 ft from the top of the tree the angle of depression of the object is found to be 30^o. Find the height of the tree.
12. The angle of elevation of a cloud from a point 60 m above a lake is 30^o and angle of depression of the reflection of cloud in the lake is 60^o. Find the height of the cloud.
13. There are two temples, one on each bank of a river, just opposite to each other. One temple is 40 m high. As observed from the top of this temple, the angles of depression

of the top and the foot of the other temple are 12^o 30' and 21^o 48' respectively. Find in metres, the width of the river and the height of the other temple approximately.

14. From an aeroplane flying horizontally above a straight road, the angles of depression of two consecutive kilometer stones on the road are observed to be 45^o and 30^o respectively. Find the height of the aeroplane above the road when the kilometer stones are (i) on the same side of the vertical through the aeroplane (ii) on the opposite sides of the vertical through the aeroplane.

15. A man standing in the field observes a flying bird in his north at an angle of elevation of 30^o and after 2.5 minutes he observes the bird in his South at an angle of elevation of 60^o. If the bird flies in straight line all along at a height of $60\sqrt{3}$ metres, find its speed.

16. If the angle of elevation of a cloud from a point h metres above a lake is θ_1 and the angle of depression of its reflection in the lake θ_2 , Prove that the height of the cloud is $\dfrac{h\ (\tan\theta_2 + \tan\theta_1)}{(\tan\theta_2 - \tan\theta_1)}$.

17. Two vertical lamp - posts of equal height stand on either side of a roadway which is 50 m wide. At a point in the roadway, between the lamp - posts, the elevationsof the tops of the lamp - posts are 60^o and 30^o. Find the height of each lamp - post and the position of this point.

18. A boy is standing on the ground and flying a kite with 100 m of string at an elevation of 30^o. Another boy is standing on the roof of a 10 m high building and is flying his kite at an elevation of 45^o. Both the boys are on opposite sides of both the kite. Find the length of the string that the second boy must have so that the two kites meet.

10. PRACTICAL GEOMETRY

INTRODUCTION

Geometry = Geo + Metry.

'*Geo*' means **Earth** and '*Metry*' means **Measure**. This is derived from Greek which means ''Measure of the Earth".This is one of the earliest branches of Mathematics.

Geometry can be broadly classified into Theoretical Geometry and Practical Geometry. The former deals with the principles of geometry by explaining the construction of figures using rough sketches, without using any geometrical instruments. On the other hand, the later deals with constructing the exact figures using geometrical instruments. We have learnt about theoretical geometry in Chapter 7. Now in this section we shall learn about practical geometry.

History of Practical Geometry :

In the 5th century BC, the Greek historian Herodotus affirmed the yearly out flow of the Nile river in Egypt and used Geometry practically to determine its boundary year after year. Egyptians recorded down some geometrical problems and their methods of solutions in 1650 BC.

On Philosophical grounds, it was said that, nature is mathematically designed and that its core is geometrically designed. Plato's assertion was that "God eternally geometrizes". Thus the Greek approach towards this field was different from Egyptians and Babylonians. The entire Greek contribution to geometry was extensive and it dominated Mathematics during1700 BC.

Likewise, the Chinese, Persians and Indians contributed much to this branch of Mathematics bringing about the birth of modern Science.

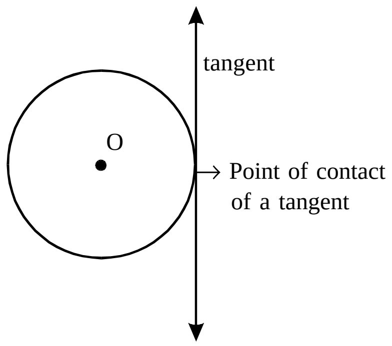
This concept is used by architects, carpenters, navigators of aeroplanes, ships and spacecrafts, photographers and others.

Review :

We have already learnt about tangent to a circle and its properties in chapter 7. Now, we are going to learn the construction of tangent to a circle. Before this, let us recall the definition for tangent and its properties.

Tangent to a circle :

A tangent to a circle is a line that touches the circle in exactly one point. The point is called the point of contact of the tangent.



Note : 1. Only one tangent can be drawn at any point on the circle.

2. Two tangents can be drawn from an external point to the circle.

10.1 CONSTRUCTION OF TANGENT TO A CIRCLE

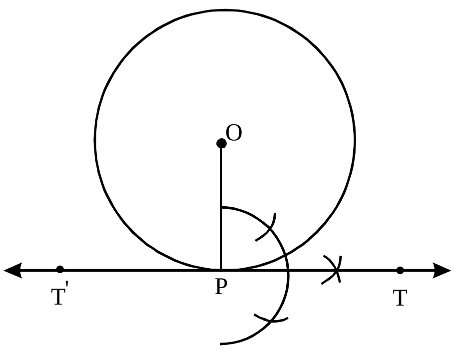
10.1.1 Introduction

Tangent to a circle at a given point can be constructed in two ways ;

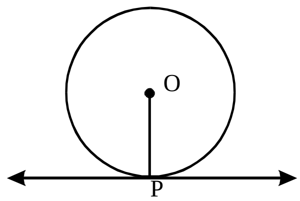
(i) By using the centre of the circle : The basic principle for this method of construction is that the tangent is perpendicular to the radius passing through the point of contact and its converse.

(ii) Without using the centre of the circle : This method of construction is based on ''Angle in alternate segment theorem''.

10.1.2 Type 1 : (a) Construction of a tangent to a circle (using its centre).



Rough Diagram



Steps of Construction :

Step 1 : Take a point O on the paper and draw a circle with centre as O and of given radius.

Step 2 : Take a point P on the circle.

Step 3 : Join OP. Here OP is the radius.

Step 4 : At P, construct an angle $\angle OPT = 90^{o}$

Step 5 : Produce TP to T' to get the line $T'PT$. $T'PT$ is the required tangent to the given circle at the point P.

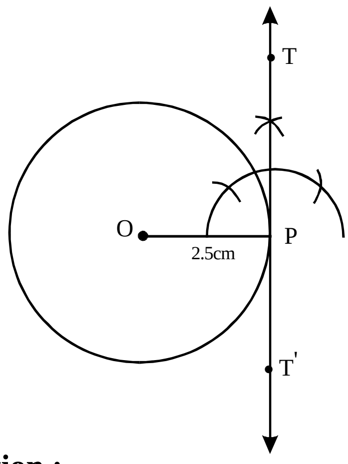
Proof : From the figure

$$\angle OPT = 90^{o}$$

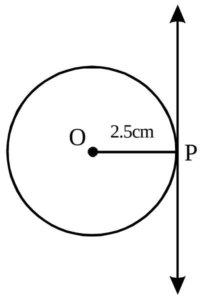
and OP is the radius of the given circle. We know that a line drawn through the end point of radius and perpendicular to it, is a tangent.

$\therefore$ $T'PT$ is a tangent to the given circle at the point P.

Example 10.1 : Draw a circle of radius 2.5 cm. Take a point P on this circle and draw a tangent at P. (using the centre)



Rough Diagram



Steps of Construction :

Step 1 : Mark a point O. With O as centre and 2.5 cm as radius draw a circle.

Step 2 : Take a point P on the circle.

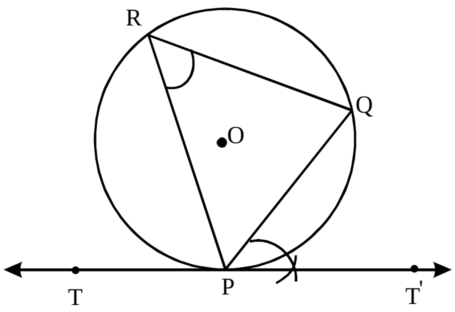
Step 3 : Join OP.

Step 4 : At P, construct an angle $\angle OPT = 90^{o}$

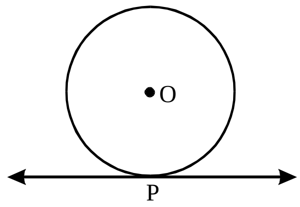
Step 5 : Produce TP to T' to get the line $T'PT$. $T'PT$ is the required tangent to the given circle at the point P.

10.1.3

Type : I (b) : Construction of a tangent to a circle at a given point on the circle, without using its centre.



Rough diagram



Steps of Construction :

Step 1 : Draw a circle with centre O and of given radius.

Step 2 : Take a point P on the circle.

Step 3 : Draw any chord PQ through the given point P on the given circle.

Step 4 : Take a point R either in major arc or in minor arc and join PR & QR.

Step 5 : Construct $\angle QPT$ equal to $\angle PRQ$ on the opposite side of the chord PQ.

Step 6 : Produce TP to T' to get the line $T'PT$. Here $T'PT$ is the required tangent to the circle at P.

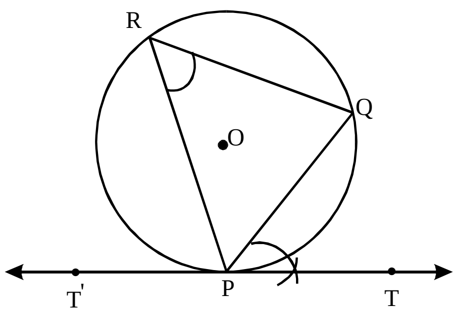
Proof : By the converse of "Angles in alternate segment" theorem.

$T'PT$ is a straight line which is drawn at the end of the chord PQ and $\angle QPT' = \angle PRQ$.

$\therefore$ $T'PT$ is a tangent to the given circle.

Following example illustrates the above procedure.

Example 10.2 : Draw a circle of radius 3 cm. At a point P on it, draw a tangent to the circle without using the centre.



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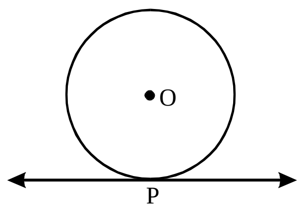


Steps of Construction :

Step 1 : Draw a circle with radius 3 cm.

Step 2 : Take a point P on the circle.

Step 3 : Through P, draw any chord PQ.

Step 4 : Mark a point R either in major arc or in minor arc.

Step 5 : Join PR and QR.

Step 6 : At P, construct ∠QPT equal to ∠ PRQ on the opposite side of the chord PQ.

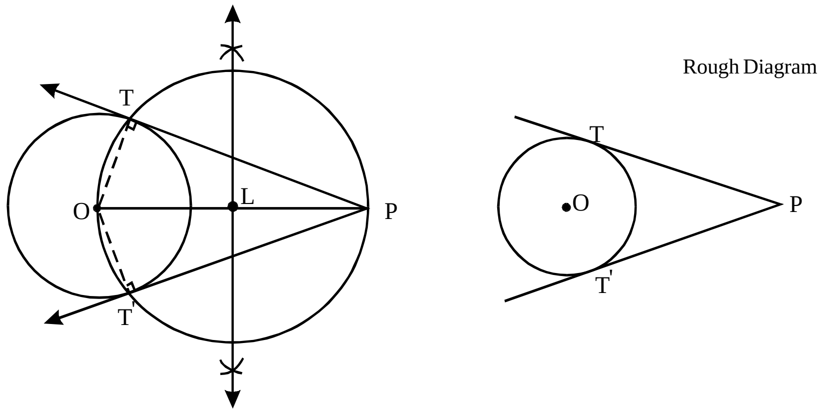
Step 7 : Produce TP to T' to get a straight line $T'PT$. The line $T'PT$ is the required tangent to the circle.

10.1.4 Construction of tangents to a circle from an external point.

Two tangents can be drawn from an external point to the circle.

Type : II (a)

10.1.5 Construction of tangents to a circle from an external point, using its centre



Steps of Construction :

Step 1 : Take a point O. With O as centre, draw a circle of given radius '*r*'

Step 2 : Mark the given external point P. Now join OP.

Step 3 : Draw perpendicular bisector of OP, intersecting OP at L.

Step 4 : With L as centre and OL = LP as radius, draw a circle to intersect the given circle at T and T'.

Step 5 : Join PT and PT$'$ to get the required tangents.

Here PT & PT$'$ are the two tangents from an external point P to the given circle.

Proof : Join OT & OT$'$ (radii of the given circle) Since OP is a diameter of the 2nd circle and angle in a semi circle is 90°.

$$\Rightarrow \angle OTP = \angle OT'P = 90^\circ$$

But OT is a radius of the given circle and $\angle OTP = 90^\circ$

$\therefore$ TP is a tangent to the circle at T.

[any line through the end of the radius and perpendicular to it, is a tangent to the circle.]

Similarly T$'$P is also a tangent to the circle at T$'$

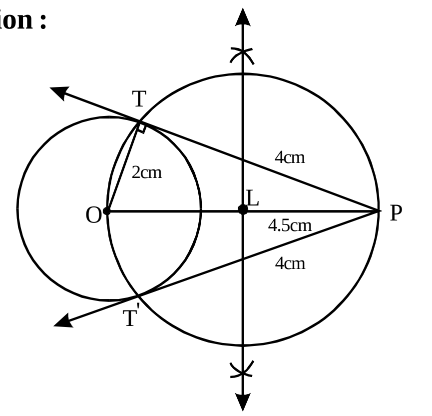
Thus PT & PT$'$ are the two tangents from P to the given circle.

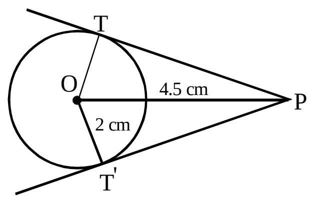
Note : The lengths of the two tangents from P are equal

(i.e.,) $PT = PT' = \sqrt{OP^2 - r^2}$ where $OT = OT' = r$ (radii of the same circle).

Example 10.3 : Draw a circle of radius 2cm. Take a point P at a distance of 4.5 cm from its centre. From P, draw two tangents to the circle (using the centre). Calculate the lengths of the tangents and verify it.

Solution :





Steps of Construction :

Step 1 : Take a point O. With O as centre and 2 cm as radius, draw a circle.

Step 2 : Mark the given external point P at a distance of 4.5 cm from its centre Now join OP.

Step 3 : Draw perpendicular bisector of OP intersecting OP at L.

Step 4 : With L as centre and OL or LP as radius, draw a circle to intersect the given circle at T and T'.

Step 5 : Join PT and PT' to get the required tangents.

Here PT and PT' are the two tangents from an external point to the given circle.

On measuring, we observe that PT = PT' = 4 cm.

Verification : In right angled ΔOTP

$$OP^2 = OT^2 + PT^2$$

$$\Rightarrow PT = \sqrt{OP^2 - OT^2} = \sqrt{(4.5)^2 - 2^2}$$

$$= \sqrt{(4.5+2)\,(4.5-2)} = \sqrt{(6.5)\,(2.5)}$$

$$PT = \sqrt{\frac{65 \times 25}{100}} \simeq \frac{8.06 \times 5}{10} \simeq 4.03 \text{ cm}$$

Similarly PT' $\simeq$ 4.03.

$\therefore$ PT = PT' = 4cm (approx)

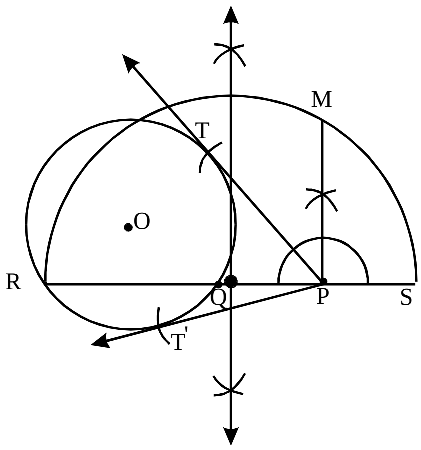
Hence verified.

Type : II (b)

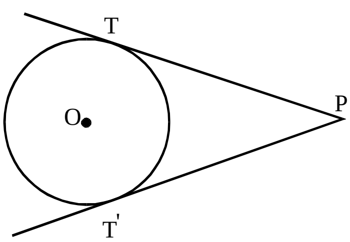
10.1.5 Construction of tangents to a circle from an external point, without using its centre.

Given : A circle and a point P outside the circle.

Required : To draw (two) tangents to the circle from P (without using its centre).



Rough diagram



Steps of Construction :

Step 1 : Draw a circle with centre O and of given radius *r*.

Step 2 : Through the given external point P, draw a secant PQR to intersect the circle at Q and R.

Step 3 : Produce QP to a point S such that QP = PS (*i.e.,*) P is the mid point of QS.

Step 4 : Draw a semi-circle with RS as diameter.

Step 5 : Draw PM $\perp$ RS , intersecting the semi circle at M.

Step 6 : With P as centre and PM as radius draw arcs to intersect the given circle at T and T'.

Step 7 : Join PT and PT'. Then PT and PT' are the required tangents.

Proof : PM is the mean proportional of RP and PS

$$\Rightarrow PM^2 = RP \times PS$$

$$\Rightarrow PT^2 = RP \times QP \quad \text{[since PM = PT ; QP = PS]}$$

$$\Rightarrow \frac{PT}{RP} = \frac{QP}{PT}$$

Now join RT and QT.

$\Rightarrow$ LP is the common angle in ΔPQT and Δ PTR.

$$\Rightarrow \Delta \text{ PQT} \ ||| \ \Delta \text{ PTR}$$

$$\Rightarrow \angle QTP = \angle PRT$$

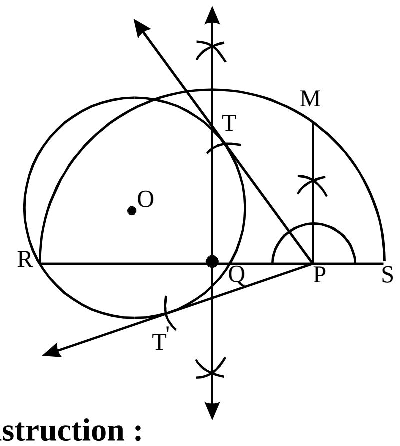
By the converse of angle in alternate segment theorem,

PT is a tangent to the given circle. Similarly PT' is also a tangent to the given circle.

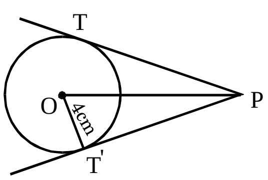
Example 10.4 : Draw a circle of radius 4 cm, Take a point P outside the circle. Without using the centre of the circle, draw two tangents to the circle from the point P. Measure the length of the tangents and verify it.

Given : A circle with centre O and radius 4 cm and an external point P.

Required : To draw two tangents to the circle from P.



Rough Diagram



Steps of Construction :

Step 1 : Draw a circle with centre O and radius 4 cm.

Step 2 : Through the given external point P, draw a secant PQR to intersect the circle at Q and R.

Step 3 : Produce QP to the point S such that QP = PS (*i.e.*,) P is the mid point of QS.

Step 4 : Draw a semi - circle with RS as diameter.

Step 5 : Draw PM $\perp$ RS, intersecting the semi circle at M.

Step 6 : With P as centre and PM as radius, draw arcs to intersect the given circle at T and T'.

Step 7 : Join PT and PT'. Then PT and PT' are the required tangents.

On measuring, we observe that PT = PT' = 4.4. cm

Verification : $PT^2 = PR \times PS$

$= 7.5 \times 2.6$

$PT = \sqrt{19.5} = 4.4$ (approx)

Similarly, PT' = 4.4 cm. (approx)

$\therefore$ PT = PT' = 4.4cm.

EXERCISE 10.1

1. Draw a circle of radius 4 cm. Take a point P on it. Using the centre of the circle, draw a tangent to the circle at the point P.

2. Draw a circle of diameter 9 cm. Take a point A on it. Using the centre of the circle, draw a tangent to the circle at the point A.

3. Draw a circle of radius 3.6 cm. Take a point P on it. Without using the centre of the circle, draw a tangent to the circle at the point P.

4. Draw a circle of diameter 8 cm. Take a point Q on it. Without using the centre of the circle, draw a tangent to the circle at the point P.

5. Draw a circle of diameter 8 cm. Take a point P at a distance of 5.5 cm from its centre. Using the centre of the circle, draw two tangents to the circle. Measure the length of the tangents.

6. Draw a circle of radius 3 cm. Take a point at a distance of 5.5 cm from the centre of the circle. From the external point draw two tangents to the circle. (using the centre of the circle) Calculate the length of the tangents and verify it.

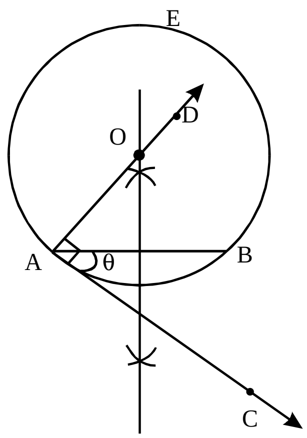
7. Draw a circle of diameter 9 cm. Take a point P outside the circle. Without using the centre of the circle draw two tangents to the circle from the point P. Calculate the length of the tangents and verify it.

8. Draw a circle of diameter 4 cm. Take a point A outside the circle. Without using the centre of the circle draw two tangents to the circle from the point A. Calculate the length of the tangents and verify it.

10.2 CONSTRUCTION OF TRIANGLES

In the lower standards, we have learnt construction of a triangle in some cases. In this section we shall learn about the construction of a triangle with base, vertical angle and altitude (or) the median on the base (or) the point on the base where the bisector of the vertical angle meets the base. We shall construct the triangle with the help of "Angle in alternate segment" theorem.

10.2.1. Construction of a segment of a circle on a given line segment containing an angle equal to a given angle.



Given : Let AB be a given line segment and θ be a given angle.

Required : To describe on AB, a segment of a circle containing the angle θ.

By using alternate segment theorem, we can draw the required segment containing an angle θ.

Steps of Construction :

Step 1 : At A, make ∠ BAC = θ

Step 2 : Draw AD ⊥ AC

Step 3 : Draw the perpendicular bisector of AB meeting AD at O.

Step 4 : With O as centre and OA as radius, draw the circle AEB

Here AEB is a required segment containing the angle θ.

Proof : Here AD ⊥ AC ⇒ ∠ CAO = 90° and OA is the radius of the circle.

⇒ AC is a tangent to the circle at A.

By alternate segment theorem.

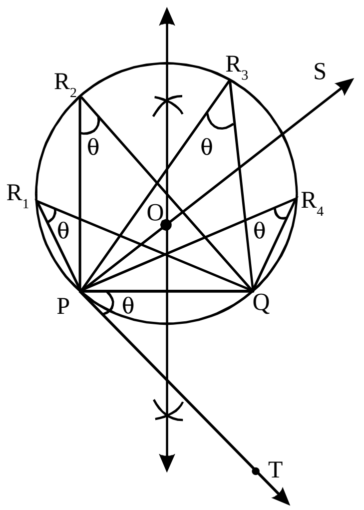
∠BAC = angle in the alternate segment AEB.

∴ The segment AEB contains an angle θ.

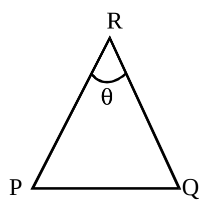
10.2.2. Construction of a triangle when its base and vertical angle are given.

Given: Base PQ, vertical angle θ.

Required : To construct a triangle PQR with base PQ and vertical angle ∠ R = θ.



Rough Diagram



Steps of Construction :

Step 1 : Draw a line segment PQ (base)

Step 2 : Below the line segment PQ, make an angle $\angle QPT = \theta$.

Step 3 : Draw a line $PS \perp PT$ at the point P.

Step 4 : Draw the perpendicular bisector of base PQ meeting the line PS at O.

Step 5 : With O as centre and OP = OQ as radius, draw a circle.

Step 6 : Take any point R_1 [or R_2 or R_3 or R_4 ...] on the circumference of the circle on the same side on which O lies.

Step 7 : Join PR_1 and QR_1.

ΔPQR_1 is the required triangle.

Remark : In the circumference of the circle on the same side on which O lies, there are infinitely many points namely R_1, R_2, R_3, ...

Consequently, infinitely many triangles [ΔPQR_1, ΔPQR_2, ΔPQR_3, ...] can be drawn with the given base and vertical angle.

Proof : ΔPQR_1 has the base PQ and $\angle PR_1Q = \theta$ [Since it is an angle in the segment which lies on the same side of O]

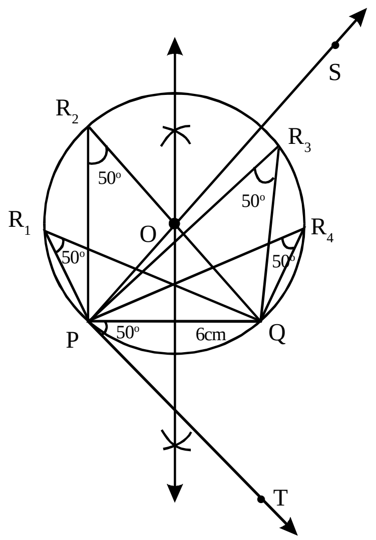
Thus, ΔPQR_1 is the required triangle.

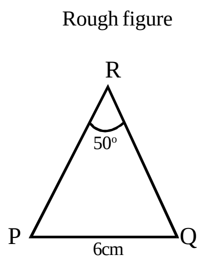
The following example illustrates the above procedure.

Example 10.5 : Construct a triangle with PQ = 6 cm and vertical angle is 50^o. How many such triangles are possible ?

Solution : Given : Base PQ = 6 cm, Vertical angle $\angle PRQ = 50^o$

Required : To construct a triangle PQR with base PQ = 6 cm vertical angle $\angle PRQ = 50^o$





Steps of Construction :

Step 1 : Draw the line segment PQ = 6 cm.

Step 2 : Below the line segment PQ, make an angle $\angle QPT = 50^o$

Step 3 : Draw a line PS $\perp$ PT at the point P.

Step 4 : Draw the perpendicular bisector of base PQ meeting the line PS at O.

Step 5 : With O as centre and OP = OQ as radius, draw a circle.

Step 6 : Take any point R_1 [or R_2 or R_3 or R_4 ...] on the circumference of the circle on the same side on which O lies.

Step 7 : Join R_1P and R_1Q

ΔPQR_1 is the required triangle.

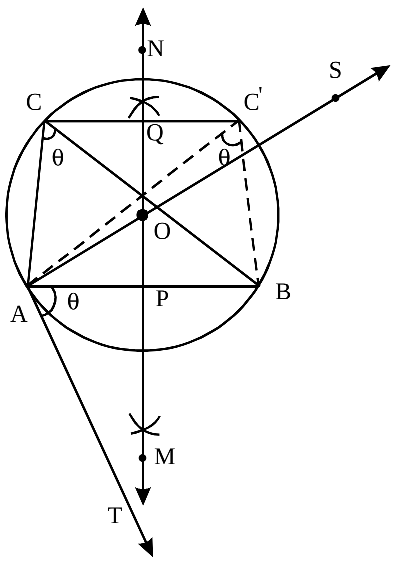
[ΔPQR_2, ΔPQR_3, ΔPQR_4.....are also the required triangles]

In this type many triangles can be drawn

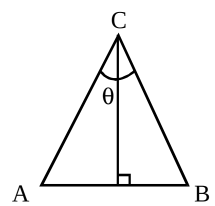
10.2.3. Construction of a triangle when the base, vertical angle and the altitude through the vertex are given.

Given : Let base be AB, vertical angle be θ , altitude be PQ.

Required : To construct a triangle ABC with base AB and vertical angle θ such that the altitude from C to AB is of length PQ.



Rough Diagram



Steps of Construction :

Step 1 : Draw a line segment AB.

Step 2 : Below AB, make an angle $\angle BAT = \theta$.

Step 3 : Draw $AS \perp AT$ at the point A.

Step 4 : Draw the perpendicular bisector MN to the base AB, meeting the line AS at O and AB at P.

Step 5 : With O as centre and OA or OB as radius, draw a circle.

Step 6 : Take a point Q on MN [PQ is the length of the altitude]

Step 7 : Through Q, draw a line parallel to AB, intersecting the circle at C and C'.

Step 8 : Join CA, CB and $C'A$, $C'B$ to obtain the triangles ABC and ABC'

Here ABC and ABC' are the required triangles.

Remark : Here the line parallel to AB through Q meets the circle atmost at two points.

$\therefore$ there are two triangles with the given conditions.

Proof : ΔABC has the base AB and $\angle ACB = \theta$ [since it is an angle in the segment which lies on the same side of O]

Altitude from C to AB is the distance between the parallel lines AB and CC'

(i.e.,) length of the altitude = PQ.

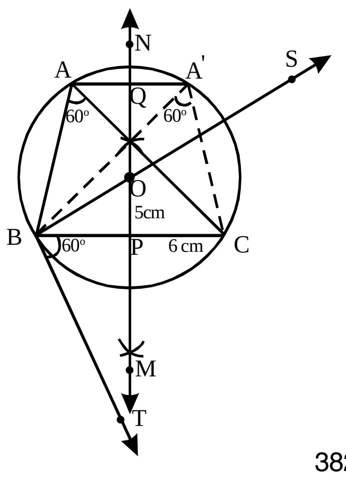
Thus ABC is the required triangle.

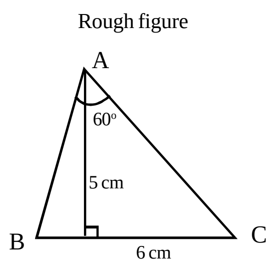
Following example will illustrate the above procedure.

Example 10.6 : Construct ΔABC in which BC = 6 cm. $\angle A = 60^{o}$ and altitude through A is 5 cm. Measure the length of median through A. Write the steps of construction. How many such triangles are possible ?

Solution : Given : BC = 6 cm, $\angle A = 60^{o}$, altitude = 5 cm.

Required : To construct a triangle ABC with base BC & vertical angle $\angle A = 60^{o}$ such that the altitude from A to BC is of length 5 cm.





Steps of Construction :

Step 1 : Draw a line segment BC = 6 cm.

Step 2 : Below BC, make an angle ∠CBT = 60° at B.

Step 3 : Draw BS ⊥ BT at the point B.

Step 4 : Draw the perpendicular bisector MN to the base BC, meeting the line BS at O and BC at P.

Step 5 : With O as centre and OB or OC as radius, draw a circle.

Step 6 : Take a point Q on MN such that PQ = 5 cm.

Step 7 : Through Q, draw a line parallel to BC, intersecting the circle at A and A'

Step 8 : Join AB, AC and A'B, A'C to obtain the triangles ABC and A'BC

Here ABC and A'BC are the required triangles.

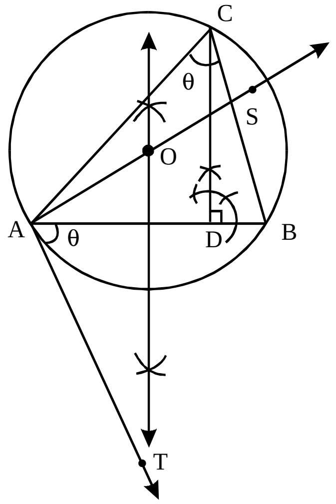
Since P is the mid point of base BC, by measuring AP, we get the length of the median AP = 5.1 cm.

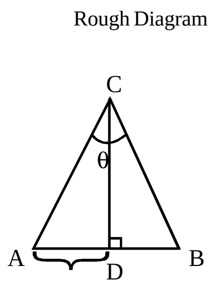
In this type two triangles are possible.

10.2.4 Construction of a triangle whose base, vertical angle and the distance of altitude from one end of the base are given.

Given : Let AB be the given base, θ be the given vertical angle andAD be the distance of altitude from the end A of base AB.

Required : To construct a triangle ABC with base AB and vertical angle θ such that the distance of altitude from A is AD.





Steps of Construction :

Step 1 : Draw a line segment AB.

Step 2 : Below AB, make an $\angle BAT = \theta$ at A

Step 3 : Draw $AS \perp AT$ at the point A.

Step 4 : Draw perpendicular bisector to base AB meeting the line AS at O.

Step 5 : With O as centre and OA = OB as radius, draw a circle.

Step 6 : Take a point D on AB. [Here AD is the distance between the altitude and the vertex A]

Step 7 : From D, draw a line perpendicular to AB meeting the circle at C.

Step 8 : Join CA and CB

$\therefore$ ABC is the required triangle.

Note : In this type, only one triangle can be drawn.

Proof :

ΔABC has the base AB and $\angle ACB = \theta$

[Since it is an angle in the segment which lies on the same side of O].

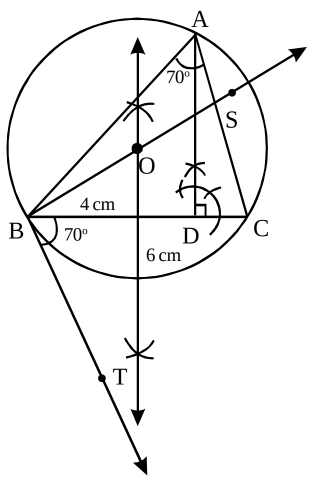
The distance between the vertex A and the altitude is AD.

Thus ABC is the required triangle.

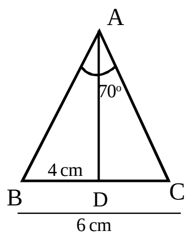
Example 10.7 **:** Construct a ΔABC in which BC = 6 cm $\angle A = 70^o$ and the foot of the perpendicular D on BC from A is 4 cm away from B.

Given : Base BC = 6 cm, Vertical angle $\angle BAC = 70^o$, BD = 4 cm.

Required : To construct a triangle ABC.



Rough Diagram



Steps of Construction :

Step 1 : Draw a line segment BC = 6 cm.

Step 2 : Below BC, make $\angle CBT = 70^o$ at B.

Step 3 : Draw BS $\perp$ BT at the point B.

Step 4 : Draw perpendicular bisector to base BC, meeting the line BS at O.

Step 5 : With O as centre and OB or OC as radius, draw a circle.

Step 6 : Take a point D on BC such that BD = 4 cm.

Step 7 : From D, draw a line perpendicular to BC meeting the circle at A.

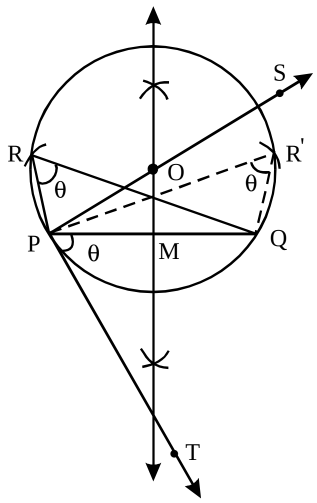
Step 8 : Join AB and AC .

ABC is the required triangle.

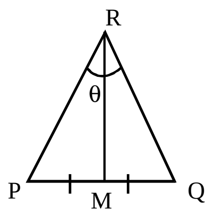
10.2.5 Construction of a triangle whose base, vertical angle and median through the vertex are given.

Given : Let Base = PQ , vertical angle $\angle PRQ = \theta$ and Length of the median = RM.

Required : To construct a triangle PQR



Rough Diagram



Steps of Construction :

Step 1 : Draw a line segment PQ.

Step 2 : Below PQ , make an $\angle QPT = \theta$ at P.

Step 3 : Draw a line PS ⊥ PT at P.

Step 4 : Draw the perpendicular bisector to PQ, meeting the lines PS at O and PQ at M.

Step 5 : With O as centre and OP = OQ as radius, draw a circle.

Step 6 : With M (mid point of PQ) as centre and RM (length of the median) as radius, draw arcs meeting the circle at R and R'.

Step 7 : Join RP, RQ and $R'P$, $R'Q$.

ΔPQR and $\Delta PQR'$ are the required triangles.

Note : In this type, two triangles can be drawn with the given conditions.

Proof : ΔPQR has the base PQ vertical angle $\angle PRQ = \theta$ [Since it is an angle in the segment which lies on the same side of O]

M is the mid point of PQ.

$\Rightarrow$ RM is the median.

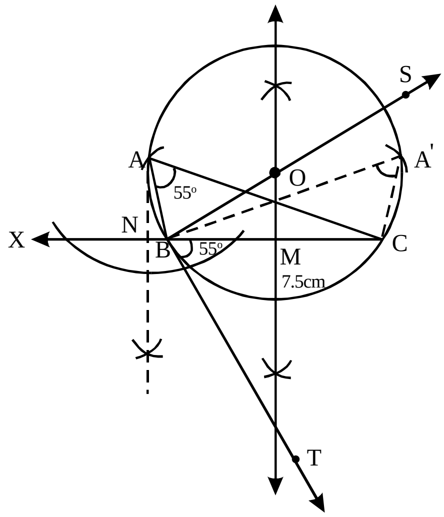
Hence PQR is the required triangle.

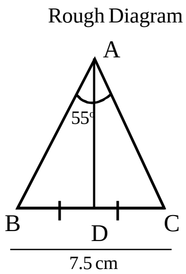
Example 10.8 : Construct a triangle ABC in which BC = 7.5 cm, $\angle A = 55^o$ and the median through A is of length 5.5 cm. Also find the length of the altitude drawn from the vertex A on BC.

Given : Base BC = 7.5 cm, Vertical angle $\angle A = 55^o$.

Length of the median AM = 5.5 cm

Required : To construct a triangle ABC.





Steps of Construction :

Step 1 : Draw a line segment BC = 7.5 cm.

Step 2 : Below BC, make $\angle CBT = 55^o$ at B.

Step 3 : Draw a line BS $\perp$ BT at B.

Step 4 : Draw the perpendicular bisector to BC meeting the lines BS at O and BC at M.

Step 5 : With O as centre and OB = OC as radius draw a circle.

Step 6 : With M (mid point of BC)as centre and 5.5 cm as radius, draw two arcs meeting the circle at A and A'.

Step 7 : Join AB, AC and A'B, A'C.

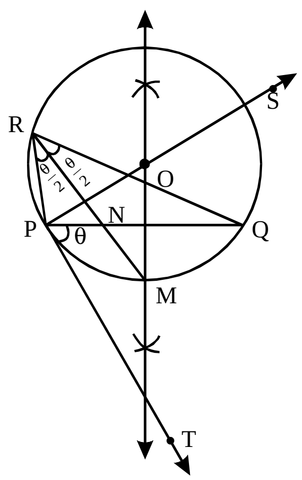
Δ ABC and ΔA'BC are the required triangles.

To find the length of the altitude from A to BC, draw AN $\perp$ XB. By measuring AN, we find that AN = 3.1 cm.

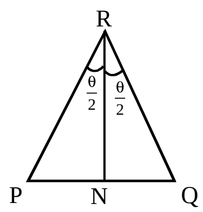
10.2.6. Construct a triangle, given the base, the vertical angle and the point on the base, where the bisector of the vertical angle meets the base.

Given : Let Base = PQ, Vertical angle $\angle PRQ = \theta$ and N be the foot of the angular bisector of $\angle PRQ$ on PQ.

Required : To construct ΔPQR.



Rough Diagram



Steps of Construction :

Step 1 : Draw a line segment PQ.

Step 2 : Below PQ, make an $\angle QPT = \theta$ at P.

Step 3 : Draw a line $PS \perp PT$ at P.

Step 4 : Draw the perpendicular bisector to PQ meeting the line PS at O.

Step 5 : With O as centre and OP = OQ as radius draw a circle. Minor arc PQ meets the perpendicular bisector of PQ at M.

Step 6 : Mark a point N on PQ. (according to the given data).

Step 7 : Join MN and produce it to meet the circle at R.

Step 8 : Join RP and RQ.

Thus ΔPQR is the required triangle.

Proof : ΔPQR has the base PQ, vertical angle $\angle PRQ = \theta$. [Since it is an angle in the segment which lies on the same side of O]

M is the mid point of minor arc PQ.

$\Rightarrow$ arc PM = arc MQ

These equal arcs subtend equal angles at R (remaining part of the circumference) $\Rightarrow \angle PRM = \angle MRQ = \frac{\theta}{2}$

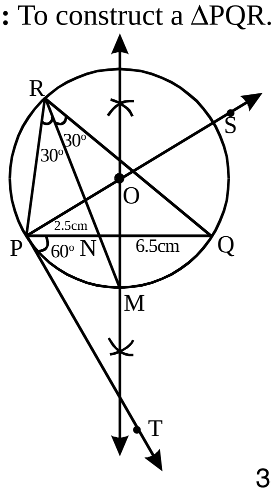
(*i.e.,*) RM is the bisector of the vertical angle meeting the base PQ at N.

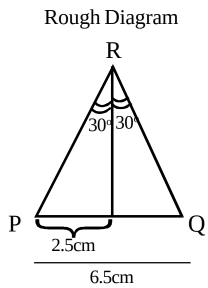
Hence PQR is the required triangle.

Example 10.9 : Construct a ΔPQR in which base PQ = 6.5.cm, vertical angle $\angle Q = 60^o$ and the bisector of the vertical angle meets the base at N where PN = 2.5 cm.

Solution : Given : Base PQ = 6.5 cm, Vertical angle $\angle Q = 60^o$ and N be the foot of the angular bisector of $\angle Q$ on PQ where PN = 2.5cm.

Required : To construct a ΔPQR.





Steps of Construction :

Step 1 : Draw a line segment PQ = 6.5 cm.

Step 2 : Below PQ, make an $\angle QPT = 60^o$ at P.

Step 3 : Draw a line PS $\perp$ PT at P.

Step 4 : Draw the perpendicular bisector to PQ meeting the line PS at O.

Step 5 : With Q as centre and OP = OQ as radius, draw a circle. Minor arc PQ meets the $\perp$ bisector of PQ at M.

Step 6 : Mark a point N on PQ such that PN = 2.5 cm.

Step 7 : Join MN and produce it to meet the circle at R

Step 8 : Join RP and RQ

Thus Δ PQR is the required triangle.

EXERCISE 10.2

Write the steps of construction for the following constructions [Proof of the construction is not required]

1. Construct a triangle with BC = 4.5 cm and vertex A such that $m\angle BAC = 55^o$. How many such triangles are possible ?
2. Construct a triangle with PQ = 7 cm and vertex R such that $m\angle PRQ = 30^o$. How many such triangles are possible ?
3. Construct a triangle with base AB = 6.5 cm and vertex C such that $\angle ACB = 45^o$.
4. Construct a ΔABC in which BC = 6.2 cm, $\angle A = 60^o$ and the altitude through A is 2.6 cm. Measure the length of median through A.
5. Construct a ΔPQR having base 5 cm, vertical angle 45^o and altitude through R is 4 cm. How many such triangles can be constructed ?
6. Construct a ΔABC such that AB = 6cm, $m\angle C = 40^o$ and altitude from C to AB is of length 4 cm. Measure the length of median through C.
7. Construct a triangle PQR such that PQ = 8 cm, $m\angle R = 120^o$ and altitude from R to PQ is 2 cm. Measure the length of the median from R.
8. Construct a ΔPQR in which base PQ = 6.5 cm, vertical angle $\angle R = 65^o$ and the foot of the perpendicular M on PQ from R is 3.5 cm away from Q. Measure the length of the altitude.
9. Construct a ΔXYZ in which base XY = 7cm, $m\angle Z = 50^o$ and the altitude is at a distance of 5 cm from X.

10. Construct a ΔABC such that AB = 5.6 cm, $m\angle C = 60^o$ and median through the vertex C is 4 cm.

11. Construct a ΔPQR in which PQ = 6cm, $m\angle R = 50^o$ and the median through R is 5.5 cm. Find the length of the altitude from A.

12. Construct a ΔPQR such that PQ = 6cm, $\angle R = 45^o$ and the median through R is 4.5 cm. Find the length of the altitude drawn from R to PQ.

13. Construct a ΔABC such that AB = 7.8 cm, $m\angle C = 40^o$ and AD = 3 cm where D is the point of intersection of the bisector of $\angle C$ on AB. Measure the length of the altitude from C.

14. Construct a ΔPQR in which PQ = 6.5 cm, vertical angle $\angle R = 130^o$ and the bisector of the vertical angle meets the base at S where PS = 2.5 cm. Measure the length of the median from R.

15. Construct a ΔABC such that AB = 7 cm, $m\angle C = 60^o$ and the bisector of $\angle C$ meets AB at a point D where AD = 2 cm. Measure the length of the median from C.

11. STATISTICS

INTRODUCTION

For a long time, ‘‘Statistics’’ referred to information about the government. The word itself comes from the Latin Statisticus, meaning ‘‘of the state’’. The term was easily transferred during the 19^{th} century to numerical information of other kinds and later to methods for analyzing that information.

But today, there are methods by which statistics obtained from a small sample can be used to represent a voluminous data. In fact, a sample as small as 1600 may be used to predict the outcome of a national election. Information gathering is conducted by a variety of people, including medical researchers, scientists, advertisers and political pollsters.

In our country a census is taken every 10 years, which is used for many purposes.

Statistics is the art and science of gathering, analyzing and making inferences (predictions) from numerical information obtained in an experiment. This numerical information is referred to as **data.** The use of statistics which was originally associated with numbers gathered for governments, has grown significantly and is now applied in all walks of life.

Measures of Central Tendency

Most people have an intuitive idea of what is meant by an ‘‘average’’.

An average is a number that is a representative of a group of data. There are atleast four different averages the mean, the median, the mode and the midrange. Each is calculated differently and may yield different results for the same set of data. Each will result in a number near the centre of the data ; for this reason, averages are commonly referred to as **Measures of Central Tendency.**

The arithmetic mean of a series of observed values is defined as the sum of the observed values divided by the number of values observed.

Arithmetic mean for ungrouped data

Arithmetic mean ($\overline{x}$) = $\dfrac{\Sigma x_i}{n}$ where $\Sigma x_i = x_1 + x_2 + x_3 + ... + x_n$.

Arithmetic mean for grouped data.

Arithmetic mean ($\bar{x}$) = $\dfrac{\Sigma fx}{\Sigma f}$

11.1 MEASURES OF DISPERSION

We have already discussed about the various measures of central tendency. Though averages are highly useful in statistical analysis, they fail to give us an idea about the extent to which the items of distribution deviate from the central value.

Consider the following data :

A : 55, 51, 46, 48

Total = 200, $\bar{x}$ = 50

B : 32, 68, 60, 40

Total = 200 $\bar{x}$ = 50

The two distributions may have the same mean and the same total frequency and yet they may differ in the extent to which the individual items differ from the central value. In the series A, the items are not very much scattered, as the minimum value of the series is 46 and the maximum value is 55. In the series B, the items are widely scattered, as the minimum value is 32 and the maximum value is 68. Hence we see the need for a study of dispersion of data.

The relative concentration or scatterness of the values of the variable around a measure of central tendency is called dispersion. This dispersion measures how the values of variable are spread about an average. There are various measures of dispersion and we consider only the following.

(1) The range (2) The standard deviation

Range

Range is the simplest method of measuring dispersion. It is the difference between the greatest and the least of the observations. But there are other and better measures of dispersion than the range. One such measure is the **standard deviation.**

Standard Deviation

This is the most important and the most powerful measure of dispersion. Its value is based upon each and every item of the data. It is denoted by the Greek letter σ (sigma).

Standard deviation is the square root of the mean of the squares of the differences of individual scores from the mean.

Standard deviation for an ungrouped data.

1. Standard deviation $\sigma = \sqrt{\frac{\Sigma d^2}{n}}$; where $d = x - \bar{x}$ and $\bar{x}$ is the mean.

2. Standard deviation $\sigma = \sqrt{\frac{\Sigma d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2}$

   where $d = x - A$ where A is the assumed mean.

Standard deviation for Discrete series

3. In discrete series when deviations are taken from Arithmetic mean.

Standard deviation $\sigma = \sqrt{\frac{\Sigma fd^2}{\Sigma f}}$; where $d = x - \bar{x}$ and $\bar{x}$ is the mean.

4. When deviations are taken from assumed mean.

   Standard deviation $\sigma = \sqrt{\frac{\Sigma fd^2}{\Sigma f} - \left(\frac{\Sigma fd}{\Sigma f}\right)^2}$

   where $d = x - A$, A being assumed mean.

Variance

The mean of the squares of the deviations of the values of the variable is called variance.

Variance $= \frac{\Sigma(x-\bar{x})^2}{n}$ and it is denoted by σ^2.

Variance is the square of standard deviation. The positive square root of the variance is defined as standard deviation.

Note :

1. The data for which standard deviation is less is said to be consistent than the one for which standard deviation is more.

2. Standard deviation of the first 'n' natural numbers is $\sqrt{\frac{n^2-1}{12}}$

Example 11.1 : Find the range of the heights of 12 girls in a class given in cm.

120, 110, 150, 100, 130, 145, 150, 100, 140, 150, 135, 125.

Solution : Range = Highest value – Lowest value

= 150 – 100 = 50

Example 11.2 : The number of lollipopes eaten by children in 5 families are 4, 6, 8, 12 and 15. Find their S.D.

Solution : Here $n = 5$

$$\text{A.M.} = \frac{\Sigma x}{n}$$

$$= \frac{4+6+8+12+15}{5} = \frac{45}{5} = 9$$

x	$d = x - \bar{x}$ $= x - 9$	d^2
4	–5	25
6	–3 } –9	9
8	–1	1
12	3	9
15	6 } 9	36
	$\Sigma d = 0$	$\Sigma d^2 = 80$

$$\text{S.D.} = \sqrt{\frac{\Sigma d^2}{n}} = \sqrt{\frac{80}{5}} = \sqrt{16} = 4$$

Example 11.3 : The scores of a cricketer in 7 matches are 70, 80, 60, 50, 40, 90, 95. Find the standard deviation.

Solution : Let the assumed mean be 70. Here $n = 7$

x	$d = x - A$ $= x - 70$		d^2
40	–30		900
50	–20	–60	400
60	–10		100
70	0		0
80	10		100
90	20	55	400
95	25		625
	$\Sigma d = -5$		$\Sigma d^2 = 2525$

$$\text{S.D. } \sigma = \sqrt{\frac{\Sigma d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2}$$

$$= \sqrt{\frac{2525}{7} - \left(\frac{-5}{7}\right)^2}$$

$$= \sqrt{\frac{2525}{7} - \frac{25}{49}}$$

$$= \sqrt{\frac{17675 - 25}{49}}$$

$$= \sqrt{\frac{17650}{49}}$$

$$= 18.98$$

Standard deviation $= 18.98$

$\frac{1}{2}$ [log 17650 – log 49]

$= 1.2782$

Antilog (1.2782)

$= 18.98$

Example 11. 4 : Find the standard deviation and variance for the following data.

x	10	15	18	20	25
f	3	2	5	8	2

Solution : Let the assumed mean be 18.

x	f	$d = x - A$ $= x - 18$	d^2	fd		fd^2
10	3	–8	64	–24		192
15	2	–3	9	–6	–30	18
18	5	0	0	0		0
20	8	2	4	16		32
25	2	7	49	14	30	98
	$\Sigma f = 20$			$\Sigma fd = 0$		$\Sigma fd^2 = 340$

$$\text{S.D.} = \sqrt{\frac{\Sigma fd^2}{\Sigma f} - \left(\frac{\Sigma fd}{\Sigma f}\right)^2} = \sqrt{\frac{340}{20} - \left(\frac{0}{20}\right)^2} = \sqrt{\frac{340}{20}}$$

$$= \sqrt{17} = 4.123$$

$$\therefore \text{ Variance } = (\sigma)^2 = (\sqrt{17})^2 = 17$$

Example 11.5 : Find the variance of the following data :

Variable	10	6	18	14	22	2
Frequency	10	7	7	15	6	5

Solution : Let the assumed mean be 14.

x	f	$d = x - A$ $d = x - 14$	d^2	fd		fd^2
2	5	–12	144	–60		720
6	7	–8	64	–56	–156	448
10	10	–4	16	–40		160
14	15	0	0	0		0
18	7	4	16	28		112
22	6	8	64	48	76	384
	$\Sigma f = 50$			$\Sigma fd = -80$		$\Sigma fd^2 = 1824$

$$\text{Variance} = \frac{\Sigma fd^2}{\Sigma f} - \left(\frac{\Sigma fd}{\Sigma f}\right)^2 = \frac{1824}{50} - \left(\frac{-80}{50}\right)^2$$

$$= \frac{1824}{50} - \frac{64}{25} = 36.48 - 2.56 = 33.92$$

Example 11.6 : Calculate the standard deviation for the following distribution.

C.I	0-10	10-20	20-30	30-40	40-50
f	3	13	21	15	5

Solution : Let the assumed mean be 25.

C.I	Midvalue x	f	$d=\frac{x-A}{c}$ $d=\frac{x-25}{10}$	d^2	fd		fd^2
0-10	5	3	–2	4	–6		12
10-20	15	13	–1	1	–13	–19	13
20-30	25	21	0	0	0		0
30-40	35	15	1	1	15	25	15
40-50	45	5	2	4	10		20
		Σf =57			Σfd = 6		Σfd^2 = 60

$$\text{S.D.} = C \times \sqrt{\frac{\Sigma fd^2}{\Sigma f} - \left(\frac{\Sigma fd}{\Sigma f}\right)^2}$$

$$= 10\sqrt{\frac{60}{57} - \left(\frac{6}{57}\right)^2}$$

$$= 10\sqrt{\frac{60}{57} - \frac{36}{3249}}$$

$$= 10\sqrt{\frac{3420-36}{3249}}$$

$$= 10\sqrt{\frac{3384}{3249}}$$

$\frac{1}{2}$ [log 3384 – log 3249]

= 0.0088

Antilog (0.0088)

= 1.021

$= 10 \times 1.021 = 10.21$

S.D. $= 10.21$

Example 11.7 : Calculate the standard deviation for the following distribution.

C.I	0-10	10-20	20-30	30-40
f	4	7	8	3

Solution : Let the assumed mean be 15.

C.I	x	f	$d = \frac{x-A}{c}$, $d = \frac{x-15}{10}$	d^2	fd	fd^2
0 - 10	5	4	–1	1	–4	4
10 - 20	15	7	0	0	0	0
20 - 30	25	8	1	1	8 } 14	8
30 - 40	35	3	2	4	6	12
		$\Sigma f = 22$			$\Sigma fd = 10$	$\Sigma fd^2 = 24$

$$\text{S.D.} = \sqrt{\frac{\Sigma fd^2}{\Sigma f} - \left(\frac{\Sigma fd}{\Sigma f}\right)^2} \times c$$

$$= \sqrt{\frac{24}{22} - \left(\frac{10}{22}\right)^2} \times 10$$

$$= \sqrt{\frac{24}{22} - \frac{100}{484}} \times 10 = \sqrt{\frac{528-100}{484}} \times 10$$

$$= \sqrt{\frac{428}{484}} \times 10$$

$$= 0.9404 \times 10$$

$$= 9.404$$

$$\text{S.D.} = 9.404$$

log 428 = 2.6314
log 484 = 2.6848

$\bar{1}.9466$ ÷ 2

$\dfrac{\bar{2} + 1.9466}{2}$

$\bar{1}.9733$

Example 11.8 : The largest value of a data is 98. If the range of the data is 78. Find the smallest value of the data.

Solution : Range = highest value – lowest value

78 = 98 – lowest value

Lowest value = 98 – 78 = 20

Example 11.9 : Find the standard deviation of the following data 30, 80, 20, 40, 50, 70, 60. Add 5 to each term and find the new S.D.

Solution : Let the assumed mean be 50. Here $n = 7$

x	$d = x - A$ $= x - 50$		d^2
20	–30		900
30	–20	–60	400
40	–10		100
50	0		0
60	10		100
70	20	60	400
80	30		900
	$\Sigma d = 0$		$\Sigma d^2 = 2800$

$$\text{S.D.} = \sqrt{\frac{\Sigma d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2}$$

$$= \sqrt{\frac{2800}{7} - \left(\frac{0}{7}\right)^2} = \sqrt{\frac{2800}{7}} = \sqrt{400} = 20$$

Then we add 5 to each value we get 25, 35, 45, 55, 65, 75, 85.

x	$d = x - A$ $= x - 55$		d^2
25	–30		900
35	–20	–60	400
45	–10		100
55	0		0
65	10		100
75	20	60	400
85	30		900
	$\Sigma d = 0$		$\Sigma d = 2800$

$$\text{S.D.} = \sqrt{\frac{\Sigma d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2}$$

$$= \sqrt{\frac{2800}{7} - \left(\frac{0}{7}\right)^2} = \sqrt{\frac{2800}{7}} = \sqrt{400} = 20.$$

From the above example we conclude ‘‘The standard deviation of a series remains unchanged when each value is added or subtracted by the same quantity.

Example 11.10 : The standard deviation of 5 values is 12. If each value is increased by 6, find the standard deviation and variance of the new set of values.

Solution : S.D. of 5 values = 12

Each value is increased by 6,

$\therefore$ New standard deviation = 12 (since S.D. is not changed by the increments in the values.)

$$\text{Variance} = (\text{S.D})^2 = (12)^2 = 144$$

$$\text{Variance} = 144$$

Note : The standard deviation of a series gets multiplied or divided by the quantity K if each value is multiplied (or) divided by K.

Example 11.11 : The variance of 5 values is 36. If each value is doubled then find the standard deviation of new values.

Solution : Variance = 36

Standard variation = $\sqrt{36}$ = 6

S.D. of 5 values = 6

When each value is doubled, S.D is also doubled

S.D. of new values = 6 × 2 = 12

EXERCISE 11.1

1. Find the S.D. of the first five natural numbers.
2. The sum of the squares of the deviations from the mean of 6 variables is 54. What is the variance ?
3. The number of ice cream cones bought by men, women, boys, girls and children on a day at the trade fair was 40, 42, 46, 48 and 44 respectively. Find the standard deviation.
4. The variance of 65 scores is 64. If each of them is divided by 2, find the standard deviation and variance of the new scores.
5. The standard deviation of 7 values is 15. If each value is decreased by 8, find the standard deviation and variance of the new set of values.
6. Find the S.D. of the following :

x	70	74	78	82	86	90
f	1	3	5	7	8	12

7. Find the S.D. of the following :

x	6	9	12	15	18
f	7	12	13	10	8

8. Find the S.D. of the following :

C.I	0-10	10-20	20-30	30-40
f	3	4	2	5

9. Find the variance of the following :

C.I	20-30	30-40	40-50	50-60
f	8	6	5	4

10. Find the standard deviation of the following :

C.I	10-19	20-29	30-39	40-49	50-59	60-69	70-79
f	5	65	222	112	53	40	3

11.2 PROBABILITY

The word 'Probability and chance are quite familiar to everyone. Many a time we come across statements like ''our cricket team is likely to win the World Cup''. I have a good chance of getting selected in the school team''. Probably I may meet the collector today.

In the above sentences, the word likely, chance, probably, etc., convey some sense of uncertainity about the occurence of some events. Ordinarily, it may appear that there cannot be any exact measurement for these uncertainities. But in Mathematics, we do have methods for calculating the degree of certainity of events in numerical values, provided certain conditions are satisfied. In order to think about and measure uncertainity, we turn to a branch of Mathematics called PROBABILITY.

Before we study the theory of probability, let us learn the definition of terms which will be frequently used.

Random Experiments and Sample Space.

When we toss (or flip) a coin once, the coin may turn up in either of the two ways - 'Head' or 'Tail'. We call Head or Tail as an outcome of this activity. The set S of all possible outcomes is given by S = {H, T} where H stands for Head and T stands for Tail. The set S is called the **sample space** (of the activity of tossing a coin). This activity (of tossing a coin) is called a **random experiment.**

Thus we define the following :

Random Experiment : Any experiment whose outcomes cannot be predicted in advance or determine in advance is a random experiment.

Trial : Each performance of the random experiment is called a trial.

Sample space : The set of all possible outcomes of a random experiment is called a sample space and is denoted by S.

Sample Point : Each element of the sample space is called a sample point.

Event : An event is a subset of a sample space.

A boy throws a die. Suppose he reports that the number appeared on the die is a square number. Can we guess what number he could get. Certainly the answer is 1 or 4 as these are the only square numbers in the sample space S = {1, 2, 3, 4, 5, 6}. Thus the set E = {1, 4} is an event of this experiment.

When a coin is tossed, getting a head is an event, getting a tail is another event. Here S = {H, T} is the sample space.

E = {H} and F = {T} are the events. Note that E and F are subsets of the sample space S.

Equally likely events : Two or more events are said to be equally likely if each one of them has an equal chance of occurring.

In tossing a coin, getting a head and getting a tail are equally likely events.

Mutually Exclusive Events : Two events A and B are said to be mutually exclusive events if the occurrence of any one of them excludes the occurrence of the other event. i.e., they cannot occur simultaneously.

In terms of sets, this means A and B are disjoint. For example, in drawing a single card from a well-shuffled standard deck of playing cards, the events, A : Card is a heart, B : Card is a diamond are mutually exclusive because a card cannot be both a heart and a diamond at the same time.

But if the events are A : Card is a heart; B : Card is a king, then A and B are not mutually exclusive because we have a king of hearts.

Similarly, if we roll a die, we will get either an 'odd number' or an 'even number'. The event A ''getting an odd number'' excludes the event B ''getting an even number'' and vice-versa. Here A and B are mutually exclusive A = {1, 3, 5}, B = {2, 4, 6} and $A \cap B = \phi$

Favourable events or cases : The number of outcomes or cases which entail the occurrence of the event in an experiment are called favourable events or favourable cases.

Example : When two dice are thrown the number of cases favourable to the event of getting a sum 5 is (1, 4), (2, 3), (3, 2), (4, 1).

There are 4 cases favourable to an event of sum = 5.

Probability :

Let A be any event and the number of outcomes of an experiment favourable to the occurence of A be '*m*' and let '*n*' be the total number of outcomes which are all equally likely. Then the probability of occurence denoted by P (A) is defined as

$$P(A) = \frac{\text{Number of favourable outcomes for A}}{\text{Total number of outcomes}} = \frac{m}{n}$$

(or) in terms of set language,

If A is a favourable event, which is a subset of the sample space S of the experiment, then the probability of A denoted by P (A) is defined by

$$P(A) = \frac{n(A)}{n(S)}$$

where n (A), n (S) are the cardinal number of sets A and S provided the members of the set S are equally likely.

Observations :

1. $0 \leq P(A) \leq 1$
2. Number of outcomes which are not favourable to the event A = $n - m$. Probability of non-occurence of A denoted by A' is given by

$$P(A') = \frac{n-m}{n} = 1 - \frac{m}{n}$$

$$= 1 - P(A)$$

$$\therefore P(A') = 1 - P(A)$$

$$\therefore P(A) + P(A') = 1$$

3. If P(A) = 0, then A is an impossible event i.e., Probability of an impossible event is 0. That is P(ϕ) = 0.
4. Probability of the sure event is 1. That is P(S) = 1. S is called sure event.

Example 11.12 : A coin is tossed three times. Write its sample space.

Solution : We denote the head by H and tail by T.

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Example 11.13 : A die is thrown twice. Find out the sample space.

Solution :

Sample space = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),
(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),
(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),
(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6),
(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6) }

Example 11.14 : From a leap year selected at random, what would be the sample space for the two odd days ?

Solution : Since the leap year has 366 days which means 52 weeks and two extra days, which are known as odd days.

Sample space = {(Sunday, Monday), (Monday, Tuesday), (Tuesday, Wednesday), (Wednesday, Thursday), (Thursday, Friday), (Friday, Saturday), (Saturday, Sunday)}

Example 11.15 : Three coins are tossed simultaneously. What is the probability of getting atleast one head ?

Solution : Sample space (S) = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

$$n\,(S) = 8.$$

Let A denote the event of getting atleast one head.

$$A = \{HHH, HHT, HTH, THH, HTT, THT, TTH\}$$

Here $n\,(A) = 7$

$$P\,(A) = \frac{n(A)}{n(S)} = \frac{7}{8}$$

Example 11.16 : A perfect die is tossed twice. Find the probability of getting a total of 9.

Solution :

$n\,(S) = 36$ (*Refer Example 11.13*)

Let A denote the event of getting a total of 9.

$$A = \{(3, 6), (4, 5), (5, 4), (6, 3)\}$$

$$n\,(A) = 4$$

$$P\,(A) = \frac{n(A)}{n(S)} = \frac{4}{36} = \frac{1}{9}$$

Example 11.17 : A number is selected at random from 1 to 100 find the probability that it is not a square number.

Solution : Sample space = {1, 2, 3, 4, ... 100}

$$n\,(S) = 100$$

Let A denote the event of getting a square numbers.

$$A = \{1, 4, 9, 16, 25, 36, 49, 64, 81, 100\}$$

$$n(A) = 10$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{10}{100} = \frac{1}{10}$$

$$P(A') = 1 - P(A)$$ [Here A' donotes the event of non-square numbers]

$$= 1 - \frac{1}{10} = \frac{9}{10}$$

∴ Probability of the number being a non-square number is $\frac{9}{10}$

Example 11.18 : A card is drawn from a well shuffled pack of 52 cards. Find the probability of (i) a red card (ii) a king, (iii) a card of club, (iv) a black ace (v) a king of heart.

Solution : Total number of cards = 52

$$n(S) = 52$$

(i) Let A denote the event of drawing a red card

$$n(A) = 26$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{26}{52} = \frac{1}{2}$$

(ii) Let B denote the event of drawing a king

$$n(B) = 4$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{4}{52} = \frac{1}{13}$$

(iii) Let C denote the event of drawing a card of club.

$$n(C) = 13$$

$$P(C) = \frac{n(C)}{n(S)} = \frac{13}{52} = \frac{1}{4}$$

(iv) Let D denote the event of drawing a black ace.

$$n(D) = 2$$

$$P(D) = \frac{n(D)}{n(S)} = \frac{2}{52} = \frac{1}{26}$$

(v) Let E denote the event of drawing a king of heart.

$$n(E) = 1$$

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{52}$$

$$P(E) = \frac{1}{52}$$

Example 11.19 : Find the probability of drawing a red ball from a bag containing 4 white and 6 black balls.

Solution : $n(S) = 4 + 6$

$$n(S) = 10$$

Let A denote the event of drawing a red ball

$$n(A) = 0$$

Because the event of drawing a red ball is an impossible event.

$$P(A) = \frac{n(A)}{n(S)} = \frac{0}{10} = 0$$

Example 11.20 : What is the probability that a number selected from the first 25 natural numbers is a prime number.

Solution : $S = \{1, 2, 3, 4, 5, ... 25\}$

$$n(S) = 25$$

Let A denote the event of getting a prime number.

$$A = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$$

$$n(A) = 9$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{9}{25}$$

Example 11.21 : A coin is tossed twice. Find the probability of getting exactly one head.

Solution : Sample space $= \{HH, TT, TH, HT\}$

$$n(S) = 4$$

Let A denote the event of getting exactly one head.

$$A = \{TH, HT\}$$

$$n(A) = 2$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{2}{4} = \frac{1}{2}$$

Example 11.22 : Find the probability of getting an even number when a die is thrown.

Solution : Sample space (S) = {1, 2, 3, 4, 5, 6}

$$n(S) = 6$$

Let A denote the event of getting an even number.

$$A = \{2, 4, 6\}$$

$$n(A) = 3$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

Example 11.23 : What is the probability that a leap year selected at random will have 53 sundays ?

Solution : A leap year contains 366 days. (i.e.,) 52 weeks and 2 additional days. The possibility for these two additional days are

Sample space = {(Sunday, Monday), (Monday, Tuesday), (Tuesday, Wednesday), (Wednesday, Thursday), (Thursday, (Friday), (Friday, Saturday), (Saturday, Sunday)}

$$n(S) = 7$$

Thus there are seven possibilities out of which 2 are favourable for getting the 53rd Sunday.

A = {(Sunday, Monday), (Saturday, Sunday)}

$$n(A) = 2$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{2}{7}$$

Example 11.24 : A two digit number is formed with the digits 3, 5, and 7. Find the probability that the number so formed is greater than 57. (Repetition of the digits not allowed)

Solution : Sample space (S) = {35, 37, 53, 57, 73, 75}

$$n(S) = 6$$

Let A denote the event of getting the number so formed is greater than 57.

$$A = \{73, 75\}$$

$$n(A) = 2$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

Example 11.25 : Two dice are thrown together. Find the probability that the two digit number formed with the two numbers turning up is divisible by 3.

Solution : [For sample space refer example 11.13]

We have, $n(S) = 36$

Let A denote the event of getting the two digits numbers formed with the two numbers turning up which are divisible by 3.

$$A = \{(1, 2), (1, 5), (2, 1), (2, 4), (3, 3), (3, 6), (4, 2), (4, 5), (5, 1), (5, 4), (6, 3), (6, 6)\}$$

$$n(A) = 12$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{12}{36} = \frac{1}{3}$$

EXERCISE 11.2

1. Find the sample space when two coins are tossed.
2. Find the sample space for an observation of a birthday of a child.
3. What is the probability that a number selected from the first 30 natural numbers in a prime number ?
4. There are 8 dozen mangoes in a box. Six mangoes are rotton. Find the probability of choosing a good mango.
5. If two dice are thrown together, find the probability that the sum of the numbers is less than 6.
6. Three coins are tossed together ; find the probability that exactly two heads turn up.
7. Two dice are thrown together. What is the probability of one number being double the other ?
8. A bag contains 5 red and 6 black balls. Find the probability that a ball drawn at random is red in colour.

9. A number is selected at random from 1 to 100. Find the probability that it is a perfect cube.
10. A coin is thrown 3 times. Find the probability that 2 heads and 1 tail turn up.
11. A card is drawn from a pack of 52 cards. Find the probability that it is a black card.
12. What is the probability that a non-leap year selected at random will contain 53 fridays ?
13. Find the probability of getting a red queen from a pack of 52 cards.
14. A natural number less than or equal to 25 is chosen. Find the probability that it is a multiple of 5.

11.3 ADDITION THEOREM ON PROBABILITY

If A and B are any two events then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$

Example 11.26 : Three coins are tossed together. Find the probability that either only 2 heads or only 2 tails turns up.

Solution : Sample space (S) = {HHH, TTT, THH, HTT, HTH, THT, HHT, TTH}

Let A denote the event of getting only 2 heads

$$A = \{THH, HTH, HHT\}$$

$$n(A) = 3$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{8}$$

Let B denote the event of getting only 2 tails.

$$B = \{HTT, THT, TTH\}$$

$$n(B) = 3$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{3}{8}$$

Both the events are mutually exclusive.

$$P(A \cup B) = P(A) + P(B)$$

$$= \frac{3}{8} + \frac{3}{8} = \frac{6}{8} = \frac{3}{4}$$

$\therefore$ Probability of getting only two heads or only two tails is $\frac{3}{4}$

Example 11.27 : Two dice are thrown together. What is the probability of getting a total 8 or a product 12.

Solution : [For sample space refer example 11.13]

We have, $n(S) = 36$

Let A denote the event of getting a total of 8.

$$A = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$$

$$n(A) = 5$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{5}{36}$$

Let B denote the event of getting a product of 12.

$$B = \{(2, 6), (3, 4), (4, 3), (6, 2)\}$$

$$n(B) = 4$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{4}{36}$$

Now $A \cap B = \{(2, 6), (6, 2)\}$

$$n(A \cap B) = 2$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{2}{36}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{5}{36} + \frac{4}{36} - \frac{2}{36} = \frac{7}{36}$$

$\therefore$ Probability of getting a total 8 or a product 12 when two is $\frac{7}{36}$

Example 11.28 : A number is selected at random from 40 to 80. Find the probability that it is divisible by 6 or 9.

Solution : Sample space (S) = {40, 41, ...80}

$$n(S) = 41$$

Let A denote the event of getting a number divisible by 6.

$$A = \{42, 48, 54, 60, 66, 72, 78\}$$

$$n(A) = 7$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{7}{41}$$

Let B denote the event of getting a number divisible by 9.

$$B = \{45, 54, 63, 72\}$$

$$n(B) = 4$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{4}{41}$$

We have, $A \cap B = \{54, 72\}$

$$n(A \cap B) = 2$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{2}{41}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{7}{41} + \frac{4}{41} - \frac{2}{41}$$

$\therefore$ Required probability is $\frac{9}{41}$

Example 11.29 : Find the probability that a card drawn out of a pack of playing cards bears the number 3 or 7.

Solution : Total number of cards = 52

Let A denote the event of drawing the number 3.

$$n(A) = 4$$

$$P(A) = \frac{4}{52}$$

Let B denote the event of drawing the number 7.

$$n(B) = 4$$

$$P(B) = \frac{4}{52}$$

Both the events are mutually exclusive.

$$P(A \cup B) = P(A) + P(B)$$

$$= \frac{4}{52} + \frac{4}{52} = \frac{8}{52} = \frac{2}{13}$$

$$P(A \cup B) = \frac{2}{13}$$

Example 11.30 : The probability that a student passes English test is $\frac{2}{3}$, the probability that he passes both English and Tamil test is $\frac{11}{15}$. The probability that he passes atleast one test is $\frac{1}{3}$. What is the probability that he passes the Tamil test?

Solution : Given $P(E) = \frac{2}{3}$, $P(E \cap T) = \frac{11}{15}$ and $P(E \cup T) = \frac{1}{3}$

Where E stands for English T stands for Tamil.

$$P(E \cup T) = \frac{1}{3}$$

$$P(E \cup T) = P(E) + P(T) - P(E \cap T)$$

$$\Rightarrow \quad \frac{1}{3} = \frac{2}{3} + P(T) - \frac{11}{15}$$

$$\Rightarrow \quad P(T) = \frac{2}{5}$$

$\therefore$ Probability that the student passes the Tamil test is $\frac{2}{5}$

EXERCISE 11.3

1. A two digit number is chosen from 1 to 100. Find the probability that it is either greater than 30 or less than 16.

2. A number is selected at random out of first 100 natural numbers. What is the probability that it is either a multiple of 11 or 13.

3. A two digit number is formed of the digits 2, 5 and 9. Find the probability that it is divisible by 2 or 5. (Repetition of the digits is not allowed).

4. A card is drawn from a well shuffled pack of 52 cards. Find the probability that it is a heart or diamond.

5. A card is drawn from a well shuffled pack of 52 cards. Find the probability that it is an ace or spade.

6. A and B are two events such that P (A) = 2/5, P (B) = 1/2, P ($A \cap B$) = 1/5, find (i) P ($A' \cap B'$) (ii) P ($A' \cup B'$).

7. Two dice are thrown. What is the probability of getting a multiple of 3 on the first die or a total of 6.

8. Two dice are rolled together. Find the probability that the two digit number formed with the numbers turning up on their faces is a multiple of 7 or 5.

9. A coin is thrown 3 times. Find the probability that atleast 1 head or exactly 2 tails turning in any order.

10. The probability that Aravindh will pass in statistics examination is $\frac{2}{3}$ and the probability that he will not pass in Mathematics examination is $\frac{5}{9}$. The probability that he will pass in atleast one of the examinations is $\frac{4}{5}$. Find the probability that he will pass in both the examinations.

11. One card is selected at random from a pack of 52 cards. Find the probability that the card selected is

    (i) a 4 , (ii) a 4 or 5, (iii) not a 4, (iv) the queen of hearts, (v) a heart, (vi) a red card, (vii) a red card or a black card, (viii) a red card and a black card, (ix) a card greater than 6 and less than 10, (x) diamond king.

12. Each individual letter of the word ‘‘accommodation’’ is placed in a piece of paper, and all 13 pieces of papers are placed in a jar. If one letter is selected at random from the jar, find the probability that (i) the letter ‘a’ is selected (ii) the letter ‘c’ is selected, (iii) a vowel is selected (iv) the letter ‘c’ or ‘m’ is selected, (v) the letter ‘t’ is selected, (vi) a consonant is selected.

13. Number that PALINDROMES read the same forward and backward For example, 40304 is a five-digit palindrome. If a single number is chosen randomly from each of the following set, find the probability that it will be palindromic. (i) the set of all two digit numbers (ii) the set of all three-digit numbers.

12. GRAPHS

12.1 QUADRATIC GRAPHS

12.1.1 Introduction

In class IX, we have learnt about the construction of graphs of straight lines represented by linear equations in one or two variables. Applying the same procedure quadratic graphs can be constructed.

However, since quadratics have curvy lines (called 'Parabola') rather than straight lines generated by linear equations, there are some additional considerations to graph it.

For graphing a straight line, two points are sufficient, though we generally plot three or more points to be on safer side. But to graph a quadratic, three points will not be sufficient. Depending upon the extension of the curve, the points may be increased.

For each value of x the equation $y = ax^2 + bx + c$ gives the corresponding value of y and we obtain the ordered pairs (x, y) of real numbers. The set of all such ordered pairs which defines the graph $y = ax^2 + bx + c$ is called quadratic graph.

12.1.2 Quadratic Polynomials

In Algebra, we have already learnt about quadratic polynomials.

A polynomial with degree 2 is called quadratic polynomial. The general form of a quadratic polynomial is $ax^2 + bx + c$, where a, b, c are real numbers such that $a \neq 0$ and x is a variable.

We will denote a quadratic polynomial $ax^2 + bx + c$ by $f(x)$

(i.e.,) $f(x) = ax^2 + bx + c$ (i.e.,) $y = ax^2 + bx + c$.

12.1.3 Value of a quadratic polynomial

Let $y = ax^2 + bx + c$ be a quadratic polynomial and let α be a real number. Then, $a\alpha^2 + b\alpha + c$ is known as the value of the quadratic polynomial $y = f(x)$ and it is denoted by $y = f(\alpha)$.

(i.e.,) $f(\alpha) = a\alpha^2 + b\alpha + c$

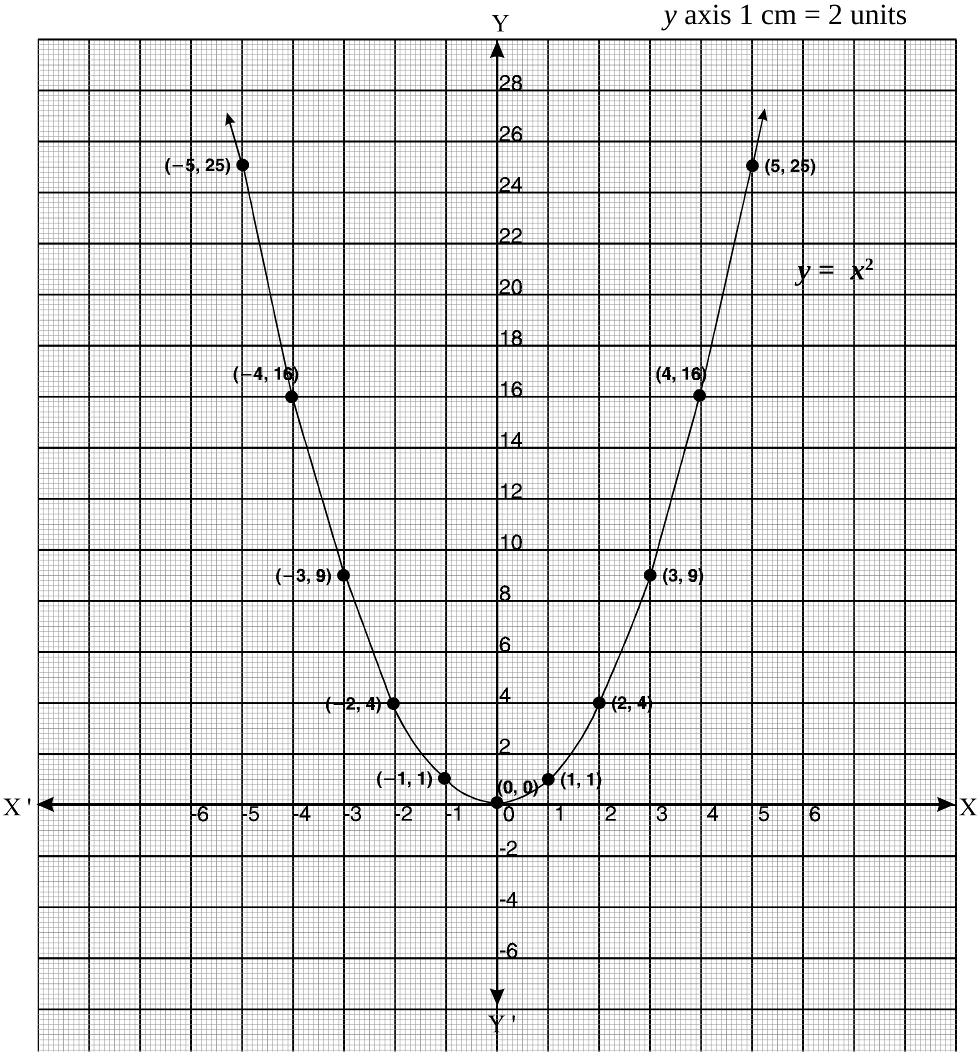
Example 12.1 : Draw the graph of $y = x^2$; $x, y \in$ R.

Solution : Assign values for x from –5 to 5 and we get the corresponding y values. Then tabulate as follows.

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
$y = x^2$	25	16	9	4	1	0	1	4	9	16	25

Plot the points (–5, 25), (–4, 16), (–3, 9), (–2, 4), (–1, 1), (0, 0), (1, 1), (2, 4), (3, 9), (4, 16), (5, 25) on the graph sheet and join the points by a smooth curve. This curve is called parabola.

Scale : x axis 1 cm = 1 unit
y axis 1 cm = 2 units



Example 12.2 : Draw the graph of $y = -x^2$

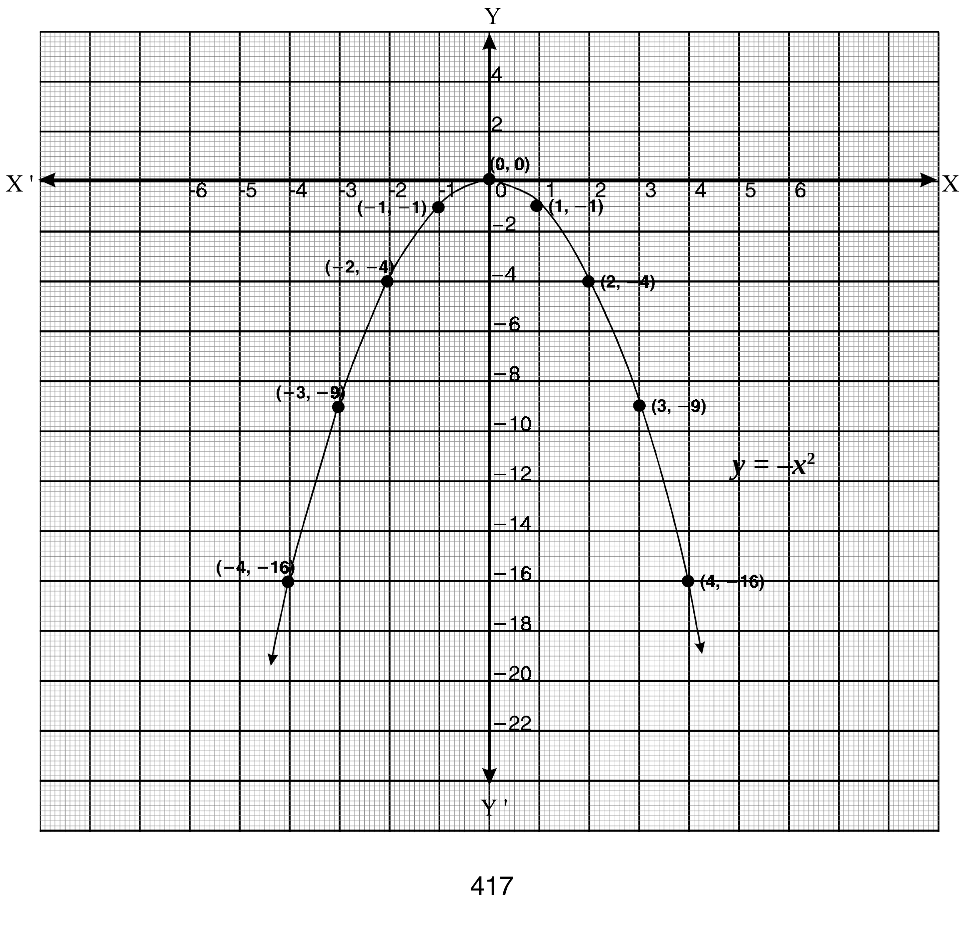
Solution : Assign values for x from –4 to 4 and get the corresponding y values. Then tabulate as follows.

$y = -x^2$

x	–4	–3	–2	–1	0	1	2	3	4
$y = -x^2$	–16	–9	–4	–1	0	–1	–4	–9	–16

Plot the points (–4, –16), (–3, –9), (–2, –4), (–1, –1), (0, 0), (1, –1), (2, –4),(3, –9), (4, –16) on the graph sheet and join the points by a smooth curve. This curve is called parabola.

Scale : x axis 1 cm = 1 unit
y axis 1 cm = 2 units



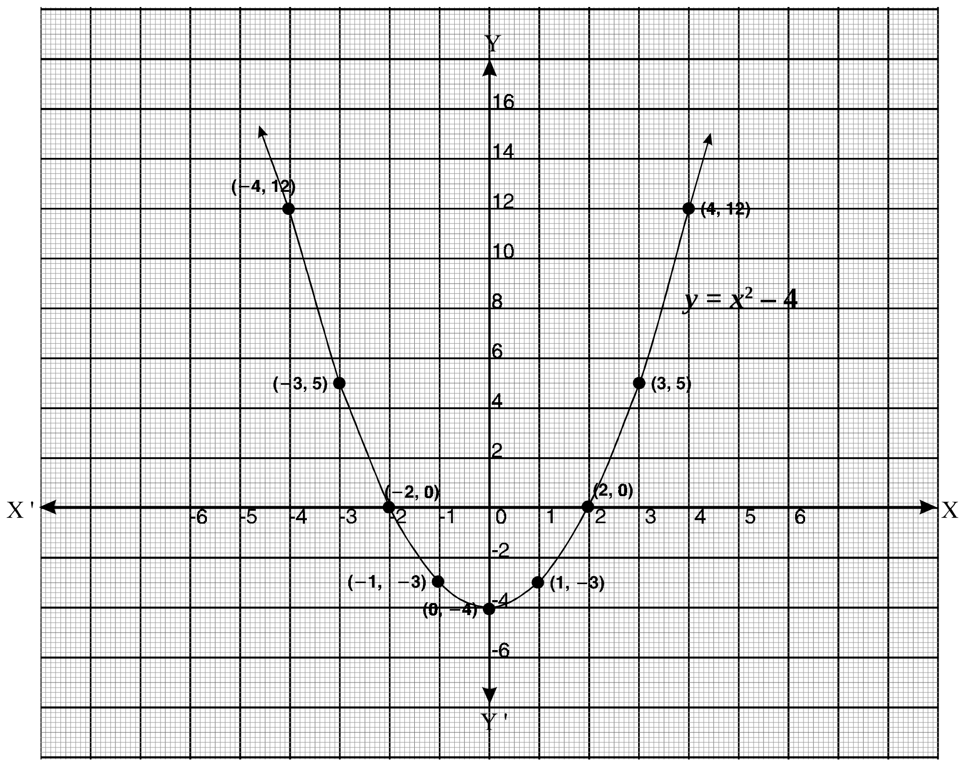
Example 12.3 : Draw the graph of $y = x^2 - 4$

Solution : Assign values for x from –4 to 4 and get the corresponding y values. Then prepare the following table.

x	– 4	–3	–2	–1	0	1	2	3	4
x^2	16	9	4	1	0	1	4	9	16
– 4	– 4	– 4	– 4	– 4	– 4	– 4	– 4	– 4	– 4
$y = x^2 - 4$	12	5	0	–3	– 4	–3	0	5	12

Plot the points (– 4, 12), (–3, 5), (–2, 0), (–1, –3), (0, – 4),(1, –3), (2, 0), (3, 5) and (4, 12) on the graph sheet and join the points by a smooth curve. We get the required graph of the Parabola $y = x^2 - 4$.

Scale : x axis 1 cm = 1 unit
y axis 1 cm = 2 units



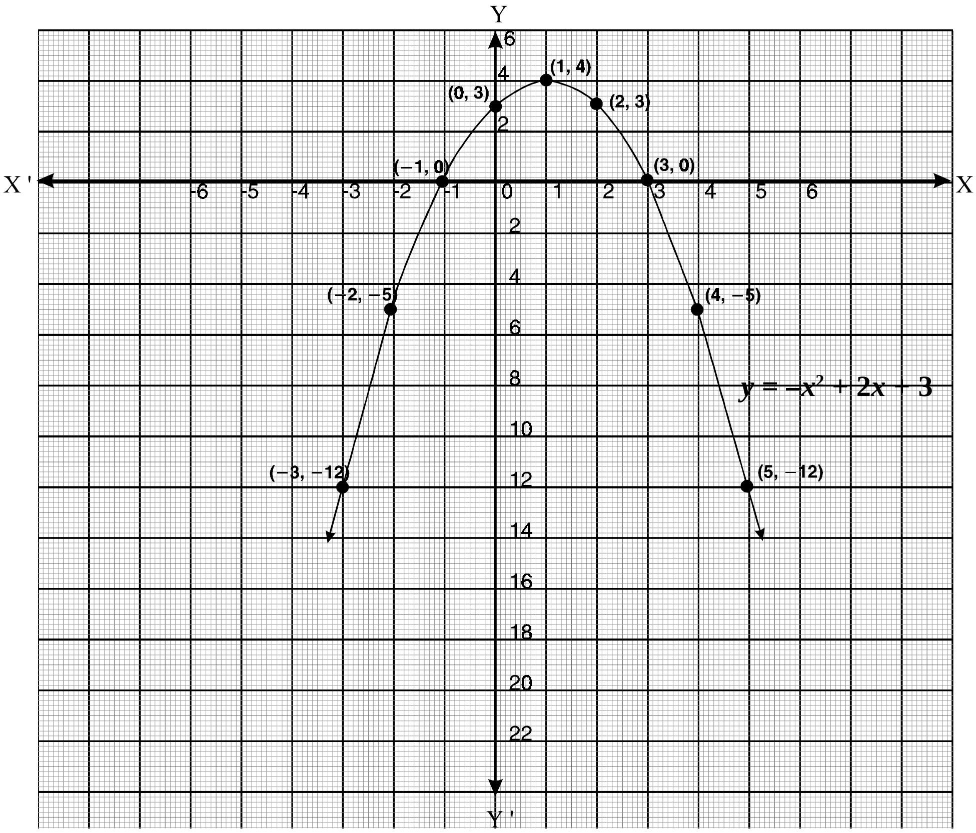
Example 12.4 : Draw the graph of $y = -x^2 + 2x + 3$

Solution : Assigning values for x from –3 to 5 we calculate the values for y and prepare the following table.

x	–3	–2	–1	0	1	2	3	4	5
$-x^2$	–9	– 4	–1	0	–1	– 4	–9	–16	–25
$+2x$	–6	– 4	–2	0	2	4	6	8	10
+3	3	3	3	3	3	3	3	3	3
$y = -x^2 + 2x + 3$	–12	–5	0	3	4	3	0	–5	–12

Plot the points (–3, –12), (–2, –5), (–1, 0), (0, 3), (1, 4),(2, 3), (3, 0), (4, –5) and (5, –12) on the graph sheet and join the points by a smooth curve. We get the required graph of the parabola $y = -x^2 + 2x + 3$.

Scale : x axis 1 cm = 1 unit
y axis 1 cm = 2 units



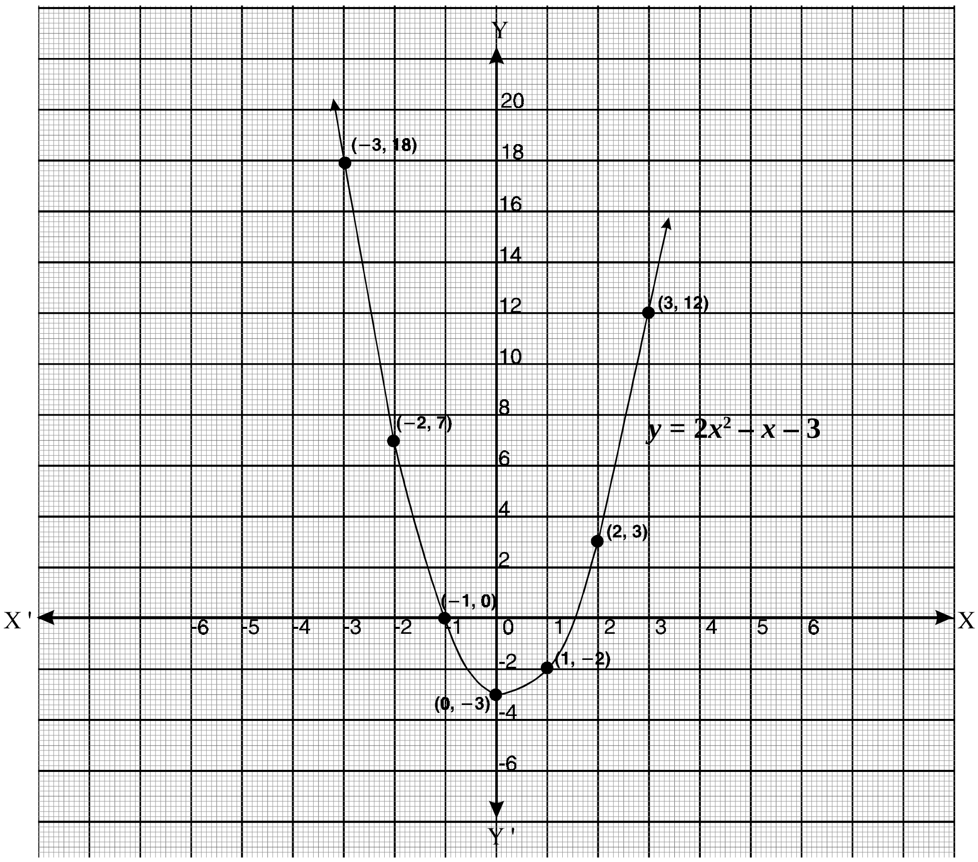
Example 12.5 : Draw the graph of $y = 2x^2 - x - 3$.

Solution : Assign values for x from –2 to 3 and calculate the corresponding values for y. Then prepare the following table.

x	–2	–1	0	1	2	3
$2x^2$	8	2	0	2	8	18
$-x$	2	1	0	–1	–2	–3
–3	–3	–3	–3	–3	–3	–3
$y = 2x^2 - x - 3$	7	0	–3	–2	3	12

Plot the points (–2, 7), (–1, 0), (0, –3), (1, –2),(2, 3) and (3, 12) on the graph sheet and join the points by a smooth curve. We get the required graph of the parabola $y = 2x^2 - x - 3$.

Scale : x axis 1 cm = 1 unit
y axis 1 cm = 2 units



12.1.4 Quadratic Equation

If $f(x)$ is a quadratic polynomial, then $f(x) = 0$ is called a quadratic equation.

The general form of a quadratic equation is $ax^2 + bx + c = 0$ where $a, b, c \in$ R & $a \neq 0$.

Roots of a quadratic equation

Let $f(x) = 0$ be a quadratic equation, $x = \alpha$ is a root of $f(x) = 0$ iff $f(\alpha) = 0$.

A quadratic equation has two roots. The two roots are

(i) real & distinct

(or) (ii) real & equal

(or) (iii) not real.

12.1.5 Solving quadratic equation by Graphical Method.

In algebra we have solved the quadratic equation $ax^2 + bx + c = 0$ by algebraic method. Now we are going to solve this quadratic equation by graphical method.

In this graphical method there are two types.

Type I :

First draw the graph of the equation $y = ax^2 + bx + c$

Here $y = 0$ is the equation of x-axis

$\therefore$ Get the points of intersection of the curve $y = ax^2 + bx + c$ with x-axis [(i.e.,) $y = 0$]

The x - coordinates of the intersecting points will give the roots of the given equation.

Type II :

Split the quadratic equation into two equations representing a parabola and a straight line.

Draw their graphs.

The x - coordinates of the points of intersection of the parabola and the straight line will give the roots of the given quadratic equation.

Example 12.6 : Solve graphically the equation $x^2 - x - 12 = 0$

Solution : First draw the graph for $y = x^2 - x - 12$

[By algebraic method the roots of $x^2 - x - 12 = 0$ are $x = 4$ or $x = -3$]

$\therefore$ Assign the values for x from -4 to 5 and get the corresponding values of y.

Then prepare the following table : $y = x^2 - x - 12$

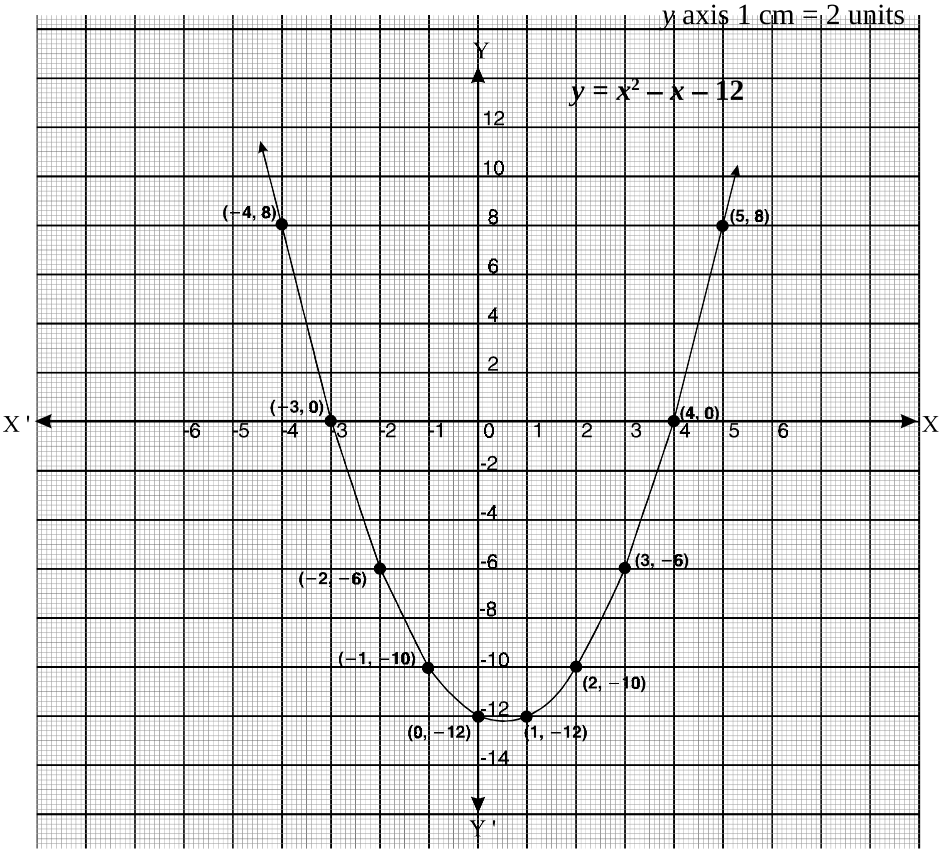
x	–4	–3	–2	–1	0	1	2	3	4	5
x^2	16	9	4	1	0	1	4	9	16	25
$-x$	4	3	2	1	0	–1	–2	–3	–4	–5
–12	–12	–12	–12	–12	–12	–12	–12	–12	–12	–12
$y = x^2 - x - 12$	8	0	–6	–10	–12	–12	–10	–6	0	8

Plot the points (–4, 8), (–3, 0), (–2, –6), (–1, –10), (0, –12), (1, –12), (2, –10), (3, –6), (4, 0), (5, 8) on the graph sheet and join the points by a smooth curve. We get the required graph of the parabola $y = x^2 - x - 12$.

From the equations $y = x^2 - x - 12$ and $x^2 - x - 12 = 0 \Rightarrow y = 0$.

But $y = 0$ is the equation of x-axis. $\therefore$ The points of intersection of the curve and the x-axis are (–3, 0) and (4, 0). Hence the solution set is {–3, 4}.

Scale : x axis 1 cm = 1 unit
y axis 1 cm = 2 units

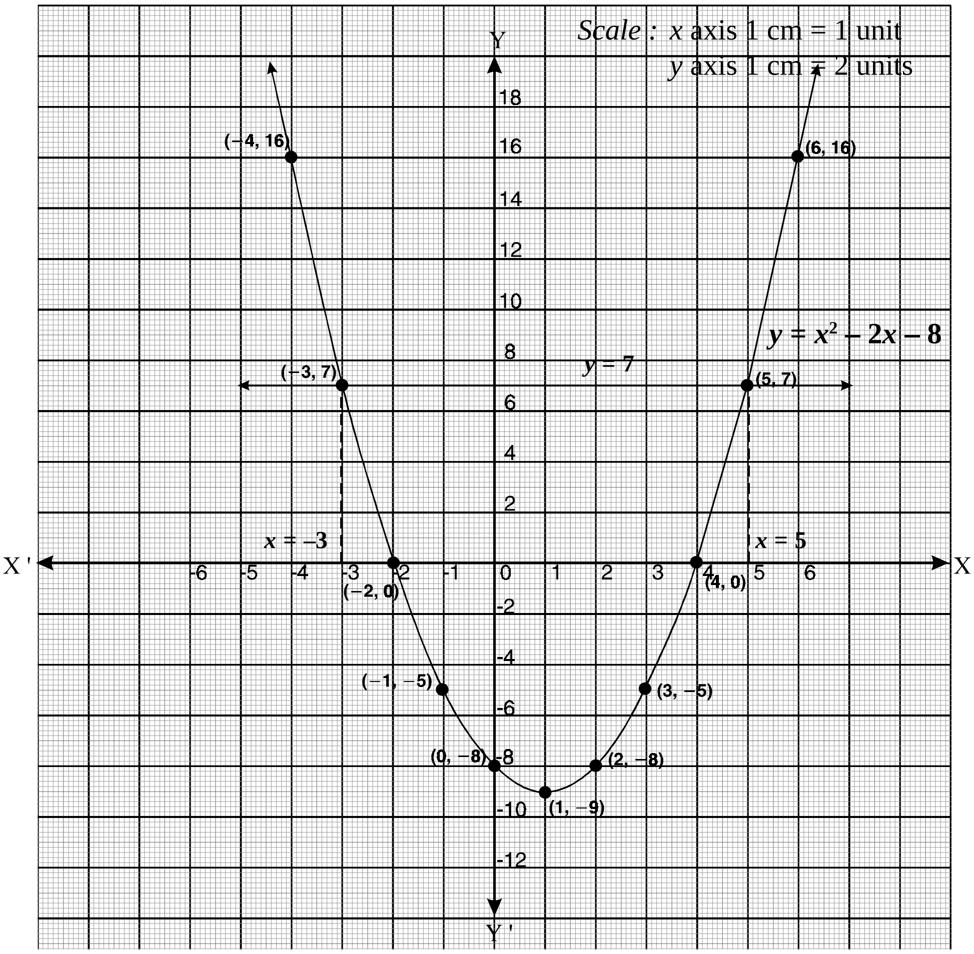


Example 12.7 : Draw the graph of $y = x^2 - 2x - 8$ and hence solve the equation $x^2 - 2x - 15 = 0$.

Solution : Assign the values for x from – 4 to 6 and get the corresponding values of y. Then prepare the following table :

$y = x^2 - 2x - 8.$

x	–4	–3	–2	–1	0	1	2	3	4	5	6
x^2	16	9	4	1	0	1	4	9	16	25	36
$-2x$	8	6	4	2	0	–2	–4	–6	–8	–10	–12
–8	–8	–8	–8	–8	–8	–8	–8	–8	–8	–8	–8
$y = x^2 - 2x - 8$	16	7	0	–5	–8	–9	–8	–5	0	7	16



Plot the points (–4, 16), (–3, 7), (–2, 0), (–1, –5), (0, –8), (1, –9), (2, –8), (3, –5), (4, 0), (5, 7) and (6, 16) on the graph sheet and join the points by a smooth curve. We get the required graph of the parabola $y = x^2 - 2x - 8$.

Now solve the equations $y = x^2 - 2x - 8$ and $x^2 - 2x - 15 = 0$ we get,

$$\begin{aligned} y &= x^2 - 2x - 8 \\ 0 &= x^2 - 2x - 15 \\ \hline y &= 7 \end{aligned}$$

which is the equation of a straight line parallel to x-axis.

Now draw the graph of $y = 7$ in the same graph sheet.

The points of intersection of curve and the line are (–3, 7) and (5, 7).

Draw perpendiculars from these two points of intersection to x-axis.

Find the x-coordinates at the foot of the perpendiculars to form the solution set.

Hence the solution set is {–3, 5}

Note : We can verify the solution using algebraic method.

Example 12.8 : Draw the graph of $y = x^2 + 2x - 3$ and hence solve the equation $x^2 - x - 6 = 0$.

Solution : First form a table for $y = x^2 + 2x - 3$

$y = x^2 + 2x - 3$

x	–4	–3	–2	–1	0	1	2	3
x^2	16	9	4	1	0	1	4	9
$2x$	–8	–6	–4	–2	0	2	4	6
–3	–3	–3	–3	–3	–3	–3	–3	–3
$y = x^2+2x-3$	5	0	–3	–4	–3	0	5	12

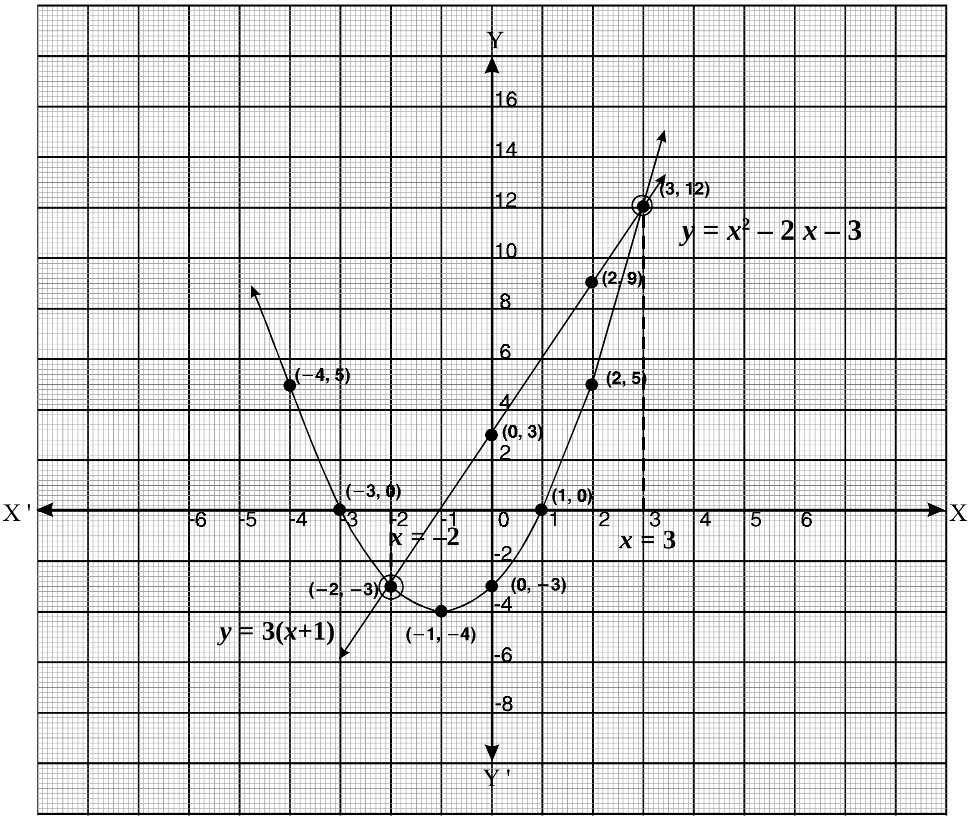
Plot the points (–4, 5), (–3, 0), (–2, –3), (–1, –4), (0, –3), (1, 0), (2, 5) and (3, 12) on the graph sheet and join the points by a smooth curve. We get the required graph of the parabola $y = x^2 + 2x - 3$.

By solving $y = x^2 + 2x - 3$ and $x^2 - x - 6 = 0$ we get,

$$\begin{aligned} y &= x^2 + 2x - 3 \\ 0 &= x^2 - x - 6 \\ \hline y &= 3x + 3 \end{aligned}$$

which is the equation of a straight line.

Scale : x axis 1 cm = 1 unit
y axis 1 cm = 2 units



Now form a table for $y = 3x + 3$ and draw the graph of it.

x	–2	0	2
$y = 3x + 3$	–3	3	9

From the graph,

The points of intersection of the parabola and the straight line are (–2, –3) and (3, 12).

Draw the perpendiculars from these two intersecting points to x-axis. The x-coordinates at the foot of the perpendiculars form the solution set.

Hence the solution set = {–2, 3}

Example 12.9 : Draw the graph of $y = x^2 + 3x + 2$ and use it to solve the equation $x^2 + 2x + 4 = 0$.

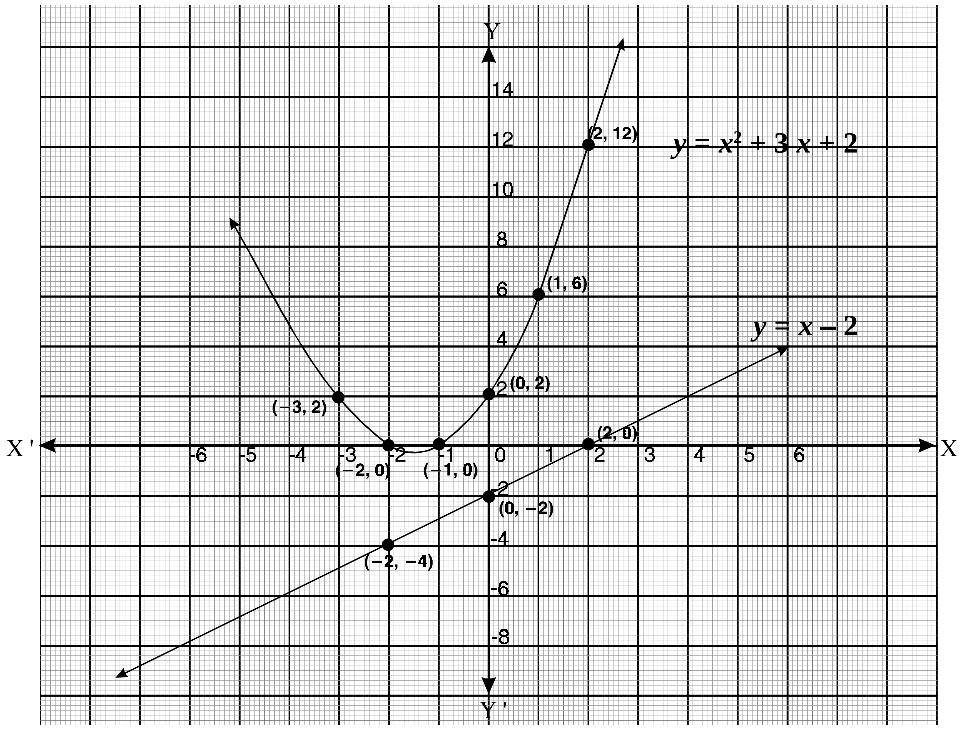
Solution : First form a table for the parabola.

$y = x^2 + 3x + 2$

x	–3	–2	–1	0	1	2
x^2	9	4	1	0	1	4
$3x$	–9	–6	–3	0	3	6
+2	2	2	2	2	2	2
$y = x^2+3x+2$	2	0	0	2	6	12

Plot the points (–3, 2), (–2, 0), (–1, 0), (0, 2), (1, 6) and (2, 12) on the graph sheet and join the points by a smooth curve. We get the required graph of the parabola $y = x^2 + 3x + 2$.

Scale : x axis 1 cm = 1 unit
y axis 1 cm = 2 units



By solving $y = x^2 + 3x + 2$. and $x^2 + 2x + 4 = 0$ we get,

$$\begin{aligned} y &= x^2 + 3x + 2 \\ 0 &= x^2 + 2x + 4 \\ \hline y &= \quad\quad x - 2 \end{aligned}$$

which is the equation of a straight line.

Now form a table for $y = x - 2$ and draw the graph of it.

x	–2	0	2
$y = x - 2$	–4	–2	0

Since the straight line does not intersect the parabola we are unable to find the x-coordinates of the point of intersection.

$\therefore$ The roots of the given equation $x^2 + 2x + 4 = 0$ are not real.

EXERCISE 12.1

1. Draw the graph of the following equations.
   (i) $y = x^2 - 5x + 6$ (ii) $y = 2x^2 + 7x - 9$ (iii) $y = 3x^2$
   (iv) $y = \frac{-5x^2}{2}$ (v) $y = (x - 2)(2x + 3)$
2. Solve graphically the following equations.
   (i) $x^2 - 4x + 3 = 0$ (ii) $3x^2 - 5x + 2 = 0$ (iii) $x - \frac{3}{x} = 2$
   (iv) $(x + 3)(x - 3) = 0$ (v) $(2x - 3)(2x + 5) = 0$
3. Solve the equation $x^2 - 2x + 1 = 0$ by using the graph of
   (i) $y = x^2 - 2x - 3$. (ii) $y = x^2 - 3x + 2$.
4. Draw the graph of $y = 2x^2 - 5x + 2$ and hence solve the equation $2x^2 - 5x + 3 = 0$.
5. Draw the graph of $y = x^2 - 3x$ and hence solve the equation $x^2 - 3x - 4 = 0$.
6. Draw the graph of $y = x^2 + 6x + 8$ and use it to solve $x^2 + 6x + 5 = 0$.
7. Draw the graph of $y = (x - 6)(x - 3)$ and hence solve the equation $x^2 - 7x + 12 = 0$. Also Verify algebraically.
8. Draw the graph of $y = x^2 - 9$ and hence solve the equation $x^2 - 2x - 8 = 0$.
9. Draw the graph of $y = x^2 + 3x + 2$ and hence solve $x^2 + x + 1 = 0$.
10. Draw the graph of $y = 2x^2 + x - 1$ and hence show that the equation $2x^2 + 5x + 5 = 0$ does not have real roots.

OBJECTIVE TYPE QUESTIONS

Choose the correct answer :

1. NUMBER WORK

1. In an A.P, if the sum of first n terms is $S_n = n^2p$ and sum of first m terms is $S_m = m^2p$, where m, n, p are positive integers and $m \neq n$, then S_p is

   a) n^2S_p b) $\frac{m^2p}{n^2}$ c) $(m^2+n^2)\,p$ d) p^3

2. If m, p, q are consecutive terms in an A.P, then p is

   a) $\frac{mq}{2}$ b) $\frac{m-q}{2}$ c) $\frac{m^2+q^2}{2}$ d) $\frac{m+q}{2}$

3. Two A.Ps have the same common difference. If the difference between their 100^{th} terms is 111222333 then the difference between their millionth terms is

   a) 123 b) 112233 c) 111222333 d) 112333

4. The number of terms in the A.P 7, 13, 19, ... , 97 is

   a) 97 b) 17 c) 16 d) 15

5. If the first term of an A.P is 5 and its 100^{th} term is –292, then its 51^{st} term is

   a) –142 b) –149 c) 155 d) –145

6. A radioactive sample decays and the remaining sample at infinite time is given by $b = 1 - \left(\frac{1}{2}+\frac{1}{4}+\ldots\ to\ \infty\right)$, then b is

   a) 0 b) 1 c) $\frac{1}{\sqrt{2}}$ d) $\frac{1}{2}$

7. If n, p, q are in G.P, then the expression for p in terms of n and q is

   a) $\frac{n}{q}$ b) $(nq)^{\frac{1}{2}}$ c) q^2n d) nq

8. The rate of a chemical reaction is directly proportional to the concentration of its reactants. For a hypothetical chemical reaction at t =0, the rate is 0.1 mole $L^{-1}s^{-1}$ and the rate increases by a factor of 3 with each second. The rate of the reaction at t = 11 seconds is

   a) 3×0.1 b) $\frac{3^{10}}{10}$ c) 0.1 d) $3 \times (0.1)^{10}$

9. The value of $11^2 + 12^2 + \ldots + 20^2$ is

   a) 3845 b) 2485 c) 2870 d) 3255

10. If $1^2 + 2^2 + ... + 10^2 = 385$, then $2^2 + 4^2 + 6^2 + ... + 20^2$ is

a) 770 b) 1150 c) 1540 d) 385×385

11. The sum of the squares of all the elements in the set
$A = \{ x / x \in N, x$ is even and $2 < x \leq 10\}$ is

a) 216 b) 220 c) 465 d) 11220

12. If sum of first 50 natural numbers is 1275 and the sum of first 50 odd numbers is 2500, then the sum of the first 50 even numbers is

a) 2550 b) 1275 c) 1725 d) 2500

13. The value of $\dfrac{1^3 + 2^3 + ... + 10^3}{1 + 2 + ... + 10}$ is

a) 45 b) 55 c) 385 d) 285

14. Water flows into a tank. The volume of water in the tank at each minute form an A.P. If the initial volume was 5 litres and becomes 6 times after 6 minutes. The speed of water increase is

a) 5 litres / min b) 6 litres / min c) 15 litres / min d) 2 litres / min

15. In an ***Ashoka chakra***, the central angle made by the smallest sector, two small sectors, three small sectors and so on are

a) in A.P b) equal
c) in G.P d) such that their summation is 360º

16. In the sequence whose $t_n = \dfrac{3n-2}{4}$; $n \in$ N the first term of the sequence is

a) $\frac{1}{4}$ b) $\frac{3}{4}$ c) $\frac{1}{2}$ d) 1

17. The number of terms in 6, 18, 54, ... ,1458 is

a) 5 b) 7 c) 8 d) 6

18. The general term of a G.P is

a) $\dfrac{a\left(1-r^n\right)}{1-r}$ b) ar^{n-1} c) $a + (n-1)d$ d) $a\,r^{n}$

19. Gravel is being dumped in the form of a conical heap. The volume of the heap doubles with each dumping. If the initial volume of the heap is V, then the volume at 21st dumping is

a) $(2^{21}) \times V$ b) $(2^{20}) \times V$ c) 2V d) V^{21}

20. If $t_1 = n$, $t_2 = n + 1$, $t_3 = n + 2$ and so on, then $t_n =$

a) n b) $2n - 1$ c) $(2n+1)$ d) $2n$

2. MENSURATION

21. A solid toy is in the form of a hemisphere surmounted by a right circular cone. Height of the cone is 1 cm. and the diameter of the base is 2 cm. If a right circular cylinder circumscribes the solid the volume of the unused space inside the cylinder is

a) 2π cc　　b) $\frac{\pi}{2}$ cc　　c) π cc　　d) π^2 cc

22. If two cylinders have their radii in the ratio 4 : 5 and heights in the ratio 5 : 6, then the ratio of their volumes is

a) 8 : 15　　b) 15 : 8　　c) 6 : 5　　d) 4 : 5

23. The relation between the volume 'V' of a sphere of radius '*r*' and its surface area '*S*' is

a) $V = \frac{2}{3} rS$　　b) $V = \frac{r}{3} S$　　c) $V = \frac{4}{3} Sr$　　d) $V = 4S$

24. A child reshapes a cone made up of China clay of height 24 cm and radius 6 cm into a sphere. The radius of the sphere is

a) 24 cm　　b) 12 cm　　c) 6 cm　　d) 48 cm

25. A solid lead sphere of radius 8 cm is made into a number of small spherical pellets each of radius 1 cm. How many such balls can be made?

a) 8　　b) 64　　c) 216　　d) 512

26. The ratio of the volume of a cube to that of a sphere which exactly fits into the cube is

a) $\pi : 1$　　b) 4 : 3　　c) $6 : \pi$　　d) $\pi : 6$

27. From a solid cylinder of height 10 cm and radius of the base 6 cm, a cone of same height and same base is removed. The volume of the remaining solid is

a) 240π cm^3　　b) 5280 cm^3　　c) 620π cm^3　　d) 360π cm^3

28. The area of cross section of a cylinder is 22 cm^2. If its height is 14 cm, then its volume in cm^3 is

a) 154　　b) 308　　c) 616　　d) 462

29. The ratio of radii and the ratio of heights of two cylinders are 1 : 4. The ratio of their volumes is

a) 1 : 16　　b) 1 : 4　　c) 64 : 1　　d) 1 : 64

30. If a rectangle of length 8 cm and breadth 4 cm is folded by bringing their breadths together to form a cylinder, then the height of the cylinder thus formed is

a) 6 cm　　b) 2 cm　　c) 8 cm　　d) 4 cm

31. A sector of a circle of radius 21cm and central angle 120° is made into a cone by bringing its radii together. Radius of the cone thus obtained is

a) 21 cm　　b) 7 cm　　c) 14 cm　　d) 10.5 cm

32. The diameter of the base of two cones are equal. If their slant heights are in the ratio 5 : 4, the ratio of their curved surface areas is

a) 5 : 4　　b) 5 : 6　　c) 4 : 5　　d) 3 : 1

33. If the radius of the base of a cone is doubled and the height is tripled, then the volume is

a) made six times b) made twelve times

c) made four times d) made two times

34. A cone and a cylinder are of same height. If the radii of their bases are in the ratio 3:5, then the ratio of their volumes is

a) 9 : 25 b) 3 : 25 c) 9 : 5 d) 25 : 3

35. The volume of a cylinder is 60 cm^3. The volume of a cone whose height and radius are same as those of the cylinder is

a) 20 cm^2 b) 20 cm^3 c) 180 cm^3 d) 60 cm^3

36. A solid sphere with total surface area 24 cm^2 is bisected into 2 hemispheres. The total surface area of any one of the hemisphere is

a) 18 cm^2 b) 6 cm^2 c) 24 cm^2 d) 36 cm^2

37. The ratio of the volumes of three solids namely cylinder, cone and hemisphere having equal base area and height is

a) 1 : 3 : 2 b) 3 : 1 : 2 c) 2 : 3 : 1 d) 1 : 2 : 3

38. The ratio of the total surface area of a solid hemisphere to the square of its radius is

a) $2\pi : 1$ b) $3\pi : 1$ c) $4\pi : 1$ d) $5\pi : 1$

3. SET LANGUAGE

39. If A, B, C are any three sets, then $B - (A \cup C) =$

a) $(A-B) \cap (A-C)$ b) $(B-A) \cap (B-C)$ c) $(B-A) \cap (A-C)$ d) $(A-B) \cap (B-C)$

40. B−A is defined as

a) $\{x/x \in A, x \notin B\}$ b) $\{x/x \in B, x \notin A\}$

c) $\{x/x \in A, x \in B\}$ d) $\{x/x \in B, x \in A$ and $B\}$

41. If $P \cup Q = \{5, 11, 14, 17, 19, 20\}$, $P \cap Q = \{14\}$ and $P = \{5, 11, 14, 17\}$, then $Q =$

a) {17, 19, 20} b) {14, 19, 17, 20} c) {5, 14, 17, 19, 20} d) {14, 19, 20}

42. Given A and B are sets and A − B = A, then

a) A and B are disjoint sets b) A and B are overlapping sets

c) B is a subset of A d) A is the subset of B

43. If A and B are any two sets, then $A \cap B' =$

a) $A + B$ b) $A - B$ c) $A \cup B$ d) $A' \cap B$

44. X-gene in a plant is responsible for red flowers and Y-gene for white flowers. If both are present, then the flowers produced are pink. If 20 plants have X-gene , 20 plants have Y-gene and there are 30 plants in all, then the total number of plants producing pink flowers are

a) 13 b) 10 c) 20 d) 30

45. Set of all even prime numbers is

a) singleton set b) null set

c) a subset of all negative integers d) infinite set

46. If {(4, 5) (5, x) } represents a constant function, then the value of x is

a) 3 b) 4 c) 5 d) 6

47. If A = {4, 9, 16, 36} and B = {1,2,3,4,5,6}, and f :A→ B defined by f = { (4, 2), (9,3), (16, 4), (36, 6)} is

a) one - one onto function b) many - one onto function

c) one - one into function d) many - one into function

48. The pre-image of 5 under the function f = {(2, –5) (3, 5) (4, –5) (5, 5)} is

a) 2 and 3 b) 3 and 4 c) 3 and 5 d) 2 and 4

49. If f : R→R defined by $f(x) = 3x - 6$ and g : R→R defined by $g(x) = 3x + k$.

If $f \circ g = g \circ f$, then k is

a) – 5 b) 5 c) 6 d) – 6

50. If $f(x) = x^2 - x$, then $f(x-1) - f(x+1)$ is

a) $4x$ b) $4x + 2$ c) $2 - 4x$ d) $4x - 2$

51. If $K(x) = 3x - 9$ and $L(x) = 7x - 10$, then $L \circ K$ is

a) $21x + 73$ b) $-21x + 73$ c) $21x - 73$ d) $22x - 73$

52. Given $f(x) = (-1)^x$ is a function from N to Z. The range of f is

a) {1} b) N c) {1, –1} d) Z

53. If $f(x) = \frac{1}{x}$ and $g(x) = -\frac{1}{x}$, then $f \circ g$ =

a) $-x$ b) $\frac{1}{x}$ c) $-\frac{1}{x}$ d) x

54. Composition of functions is

a) Associative b) Commutative

c) Commutative and Associative d) not Associative

55. Identify the wrong statement(s)

1) set of whole numbers is a subset of set of real numbers

2) $f : R \to R$ defined by $f(x) = -x$ is one - one and into

3) $f : Z \to R$ defined as $f(x) = \frac{1}{x}$ is not a function

4) B – A = B if A and B are disjoint sets

a) 2 and 3 b) 2 only c) 3 only d) 2 and 4

56. A comet is heading for Jupiter with acceleration $a = 50$ kms^{-2}. The velocity of the comet at time 't' is given by $f(t) = at^2 - at + 1$. Then the velocity at time $t = 5$ seconds is

a) 900 kms^{-1} b) 1001 kms^{-1} c) 2001 kms^{-1} d) 50 kms^{-1}

4. CONSUMER ARITHMETIC

57. Vignesh deposits Rs. 100 every month as R.D for 2½ years. The amount deposited by Vignesh will be

a) Rs. 3600 b) Rs. 3000 c) Rs. 4800 d) Rs. 3500

58. A recurring deposit of Rs. 50 per month at 10% S.I per annum will fetch at the end of 2 years an interest of

a) Rs. 250 b) Rs. 125 c) Rs. 375 d) Rs. 500

59. Ram deposits Rs.500 p.m. in R.D for 6 years in a bank which pays 10% S.I per annum. The effective period for the R.D in years is

a) 6 b) 21 c) 216 d) 219

60. A recurring deposit of Rs. 80 per month for 5 years at 8% simple interest per year yields an interest of

a) 80×5 b) $80 \times \frac{60\times61}{2\times12} \times \frac{8}{100}$ c) $80 \times \frac{69}{12} \times \frac{8}{100}$ d) $80 \times 5 \times 12 \times \frac{8}{100}$

61. A bank pays Rs. 1800 to Prathik as the maturity value at the end of 2 years at some rate of interest. If Prathik pays a monthly deposit of Rs. 50 for 2 years, then the interest paid by the bank is

a) Rs. 800 b) Rs. 100 c) Rs. 600 d) Rs. 300

62. What is the half yearly interest received for Rs. 25000 in a bank on a fixed deposit for 2 years, if the rate of interest is 10%?

a) Rs. 2500 b) Rs. 1250 c) Rs. 3750 d) Rs. 5000

63. Find the quarterly interest due on Rs.1000 at 12% rate of interest.

a) Rs. 120 b) Rs. 40 c) Rs. 30 d) Rs. 60

64. The difference between C.I and S.I on Rs. 6000 for 2 years at 4% per annum is

a) Rs. 19.2 b) Rs. 9.6 c) Rs. 4.8 d) Rs. 12.4

65. If Raja deposits Rs. 1200 for 2 years and receives Rs. 60 as interest at the end of 6 months, then the rate of interest is

a) 5% b) 10% c) 4% d) 6%

66. If the difference between C.I and S.I on Rs. 2000 for 2 years is Rs. 20, then the rate of interest is

a) 10% b) 20% c) 30% d) 15%

5. ALGEBRA

67. H.C.F. of (x^2+1) and (x^4+1) is

a) (x^2+1) b) $(x+1)$ c) 1 d) 0

68. The H.C.F. of $x^2-y^2, x^3-y^3, ..., x^n-y^n$ where $n \in$ N is

a) $x-y$ b) $x + y$ c) x^n-y^n d) 1

69. Which of the following is correct

(i) every polynomial has finite number of multiples.

(ii) LCM of two polynomials of degree '2' may be a constant

(iii) HCF of 2 polynomials may be a constant

(iv) degree of HCF of two polynomials is always less than degree of L.C.M.

a) (i) and (iii) b) (iii) and (iv) c) (iii) only d) (iv) only

70. Sum of two natural numbers is 8 and the sum of their reciprocal is $\frac{8}{15}$, then the numbers are

a) 3 and 5 b) 6 and 2 c) 7 and 1 d) 4 and 4

71. If $4x + 5y = 83$ and $\frac{3x}{2y} = \frac{21}{22}$, then $y - x = ?$

a) 3 b) 4 c) 7 d) 11

72. Degree of the polynomial $x^3 + 3x + x^6 + 2x^4$ is

a) 6 b) 3 c) 2 d) 1

73. When $f(x)$ is divided by $ax + b$, the remainder is

a) $f\left(\frac{a}{b}\right)$ b) $f\left(\frac{-a}{b}\right)$ c) $f\left(\frac{b}{a}\right)$ d) $f\left(\frac{-b}{a}\right)$

74. If $p(x)$ is divided by an expression, the remainder is $p(3)$. Then the divisor is

a) $x + 3$ b) $x - 3$ c) $p(x - 3)$ d) $p(x + 3)$

75. If $3x + 2$ is a factor of $p(x)$, then

a) $p\left(\frac{2}{3}\right) = 0$ b) $p\left(\frac{3}{2}\right) = 0$ c) $p\left(\frac{-2}{3}\right) = 0$ d) $p\left(\frac{-3}{2}\right) = 0$

76. If $p(x) = 5x^2 + 3x - 1$ and $p\left(\frac{-1}{2}\right) = 0$, then by factor theorem the corresponding factor of $p(x)$ is

a) $x - \frac{1}{2}$ b) $2x + 1$ c) $2x - 1$ d) $x + 1$

77. If $x - 1$ is a factor of $3x^2 - 2x - p$, then the value of p is

a) 0 b) –1 c) 1 d) 2

78. The polynomial with $2x + 1$ as one of the factors is

a) $2x^2 + 5x + 3$ b) $4x^2 + 6x + 9$ c) $2x^2 + x - 3$ d) $4x^2 + 4x + 1$

79. The quotient when $2x^3 + x^2 - 5x + 2$ is divided by $x + 2$ is

a) $2x^2 + 3x + 1$ b) $2x^2 - 4x + 1$ c) $2x^2 - 3x - 1$ d) $2x^2 - 3x + 1$

80. The H.C.F of $x^2 - 4$ and $x + 2$ is

a) $x^2 - 4$ b) $x + 2$ c) $x - 2$ d) $x + 3$

81. The G.C.D of $4(x-1)^2(x + 2)$ and $6(x-1)(x-2)$ is

a) $24(x-1)^2(x^2-1)$ b) $x - 1$ c) $2(x-1)$ d) $24(x-1)$

82. The L.C.M of $(x^3 - 1)$ and $(x-1)^3$ is

a) $x^3 - 1$ b) $(x-1)^3$ c) $(x^3 - 1)(x-1)^3$ d) $(x-1)^3(x^2+x+1)$

83. The L.C.M $(x+1)^2(x-3)$ and $(x^2 - 9)(x+1)$ is

a) $(x+1)^3(x^2-9)$ b) $(x + 1)^2(x^2 - 9)$ c) $(x+1)^2(x-3)$ d) $(x-9)(x+1)$

84. The L.C.M of 2^k, 2^{k+1}, 2^{k+5} where $k \in N$ is

a) 2 b) 2^k c) 2^{k+1} d) 2^{k+5}

85. If the product of two monomials is $72x^5$ and their G.C.D is $6x^2$, then their L.C.M is

a) $6x^2$ b) $12x^3$ c) $12x^3$ d) $72x^5$

86. $\dfrac{x^3 - 3x^2 + 3x - 1}{(x-1)^2} =$

a) $x + 1$ b) $x^2 + x + 1$ c) $3x^2 + 3x - 1$ d) $x - 1$

87. $\dfrac{a^2}{a^2 - b^2} + \dfrac{b^2}{b^2 - a^2} =$

a) $a - b$ b) $a + b$ c) $a^2 - b^2$ d) 1

88. $\dfrac{x+3}{x^2 - x - 6} \equiv \dfrac{A}{x+2} +$ ________

a) $\dfrac{A}{x+3}$ b) $\dfrac{B}{x+3}$ c) $\dfrac{B}{x-3}$ d) $\dfrac{B}{x^2 - x - 6}$

89. The partial fraction representation of $\dfrac{x}{(x+1)^2}$ is

a) $\dfrac{A}{x+1}$ b) $\dfrac{A}{x+1} + \dfrac{B}{(x+1)^2}$ c) $\dfrac{Ax+B}{(x+1)^2}$ d) $\dfrac{Ax+B}{(x+1)} + \dfrac{Cx+D}{(x+1)^2}$

90. If α and β are the roots of the equation $ax^2 + bx + c = 0$, then $(\alpha + \beta)^2$ is

a) $\dfrac{-b^2}{a^2}$ b) $\dfrac{c^2}{a^2}$ c) $\dfrac{b^2}{a^2}$ d) $\dfrac{bc}{a}$

91. If one root of the equation is negative of the other in the equation $ax^2+bx+c = 0$, then

a) $c = 0$ b) $a = 0$ c) $b = 0$ d) $a = 0$ and $c = 0$

92. If one root of the equation is the reciprocal of the other root in $ax^2 + bx + c = 0$, then

a) $a = c$ b) $a = b$ c) $b = c$ d) $c = 0$

93. The square root of $9x^2 + 30xy + 25y^2$ is

a) $3x+5y$ b) $3x-5y$ c) $-3x+5y$ d) $9x+25y$

94. If α and β are the roots of $ax^2 + 3x + 2 = 0$; $a < 0$. If $m = \dfrac{\alpha^2}{\beta} + \dfrac{\beta^2}{\alpha}$, then the correct statement is

a) $m < 0$ b) $m > 1$ c) $m > 0$ d) $m > 2$

95. The roots of the equation $x^2 - bx + c = 0$ are two consecutive integers then $b^2 - 4c$ =

a) 3 b) -2 c) -1 d) 1

96. If α and β are the roots of the equation $x^2 + 2x + 8 = 0$, then the value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ is

a) $\frac{1}{2}$ b) 6 c) $-\frac{3}{2}$ d) $\frac{3}{2}$

97. If α and β are the roots of the equation $x^2 - 7x + 8 = 0$, then the value of $\frac{1}{\alpha} + \frac{1}{\beta}$ is

a) $\frac{8}{7}$ b) 7 c) 8 d) $\frac{7}{8}$

98. The roots of the equation $x^2 - 8x + 12 = 0$ are

a) real and irrational b) real and rational c) real and equal d) unreal

99. The nature of the roots of $x^2 + ax + bx + ab = 0$ is

a) real, distinct and rational b) real, distinct and irrational

c) not real d) real and equal

6. MATRICES

100. If A is $(m \times n)$ matrix and B is $(n \times p)$ matrix, where m, n, p are distinct natural numbers then BA is

a) $(m \times p)$ matrix b) $(n \times n)$ matrix c) not possible d) $(p \times m)$ matrix

101. If $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, then A is

a) square matrix b) diagonal matrix c) unit matrix d) rectangular matrix

102. If $\begin{pmatrix} x+y & x-y \\ 7 & 6 \end{pmatrix} = \begin{pmatrix} 10 & 2 \\ 7 & z \end{pmatrix}$, then x, y, z are

a) 4, 6, 6 b) 6, 4, 6 c) 6, 6, 4 d) 4, 4, 6

103. If $\begin{pmatrix} 3 & 2 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 18 \end{pmatrix}$, then x and y are

a) (4, 2) b) $(-4, 2)$ c) $(4, -2)$ d) (4, 4)

104. If A = $\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ The correct statement(s) about matrix A^2 is/are

(i) A^2 is a null matrix (ii) $A^2 = I$ (iii) $A^2 + A = 0$ (iv) $A^2 = -I$

a) (ii) only b) (iv) only c) both (i) and (iii) d) both (ii) and (iv)

105. If A and B are two matrices which satisfies $A + B = B$, then A is

a) row matrix b) column matrix c) null matrix d) diagonal matrix

106. If A and B are any matrices of same order, then the relation between $(A - B)$ and $(B - A)$ is

a) $(A-B) = (B-A)$ b) $(B-A) = -(A-B)$ c) $(A-B)+(B-A)=A$ d) $(B-A)-(A+B) = 0$

107. The order of matrix A is 5×3 . The order of matrix B is 3×5. The order of $B\times A$ is

a) 5×5 b) 3×3 c) 1×1 d) 5×3

108. If $(1 \quad 2 \quad 3 \quad 4)\ X = (6)$, then the order of X is

a) 1×4 b) 4×1 c) 4×4 d) 1×1

109. State which of the following is not true?

a) Scalar matrix is a square matrix b) A diagonal matrix is a square matrix

c) A scalar matrix is a diagonal matrix d) All diagonal matrices are scalar

110. If $P = \begin{pmatrix} 6 & 7 & 8 & 9 \\ -5 & -4 & 3 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} -2 & 5 & 3 & 0 \\ 5 & 4 & -3 & 2 \end{pmatrix}$, then the order of $P-Q$ is

a) 4×4 b) not defined c) 2×4 d) 4×2

111. If $(-1 \quad -2 \quad 4) \begin{pmatrix} 2 \\ a \\ -3 \end{pmatrix} = (-10)$, then the value of a is

a) 2 b) -4 c) 4 d) -2

112. Determine the matrix A given by $(a_{ij})_{2\times2}$ if $a_{ij} = i - j$

a) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ d) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

113. If $X = (-1 \quad 3)$ and $Y = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, then $X + Y =$

a) $(0, 0)$ b) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ c) (-10) d) not defined

114. If $\begin{pmatrix} 4 & 8 \\ x & 4 \end{pmatrix} = 4 \begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix}$, then the value of x =

a) 16 b) 8 c) 4 d) 1

115. $A = \begin{pmatrix} -5 & 2 & 1 \\ 0 & -3 & 5 \\ 0 & 0 & 1 \end{pmatrix}$ is an example for

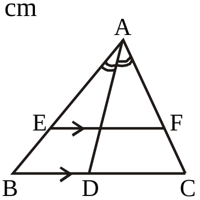
a) lower triangular matrix b) upper triangular matrix
c) diagonal matrix d) scalar matrix

7. THEORETICAL GEOMETRY

116. Two chords AB and CD of a circle intersect externally at P ; If AP = 10 cm, CP = 6 cm and PD = 5 cm, find PB.

a) 10 cm b) 3 cm c) 5 cm d) 6 cm

117. From the figure the appropriate condition is



a) $BC \times AC = \frac{EF}{BE}$ b) $BD \times AF = AE \times DC$
c) $BE \times EF = AD \times FA$ d) $BD = DC$

118. In ΔABC, AD is a median and also bisects $\angle$A. If AB =16cm, BC =8cm, then AC=

a) $\left(\sqrt{4}\right)^2$ b) 8 c) 4^2 d) 2

119. If two circles are such that their distance between the centres is greater than the sum of their radii, then the number of common tangents that can be drawn to the two circles is/are

a) 2 b) 1 c) 3 d) 4

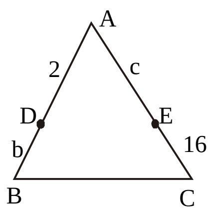
120. AB is a line segment of length 6 cm and M is its mid point. Semicircles are drawn with AM, MB and AB as diameter, all on the same side of AB. A circle is drawn to touch all the semicircles. The radius of the circle is

a) 6 b) 3 c) 2 d) 1

121. Two circles with radii 4 cm and 9 cm touch each other externally. Let R be the radius of the circle which touches these two circles as well as a common tangent to the circles. The R is

a) $\frac{36}{25}$ b) $\frac{6}{13}$ c) $\frac{9}{13}$ d) $\frac{13}{4}$

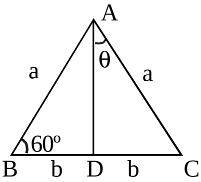
122. From the figure, 2, b, c, 16 are in G.P and their summation is 32. In the figure,



a) DE is parallel to BC b) DE is not parallel to BC
c) It does not obey BPT d) Data is insufficient

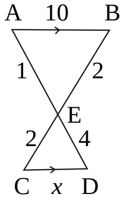
123. In the figure θ is

a) 60° b) 45°
c) 30° d) 15°



124. From the figure x is

a) 20 b) 5
c) 4 d) 10

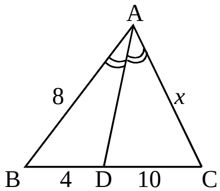


125. If the ratio of altitude of two similar triangles is 4 : 5, then the ratio of their area is

a) 1 : 2 b) 16 : 25 c) 4 : 5 d) 5 : 4

126. If AD is the bisector of $\angle A$, then AC = ?

a) 16 b) 20 c) 12 d) 18



127. In triangles ABC and DEF, $\angle A = \angle E$ and $\angle B = \angle F$. Then AB : AC is

a) DE : DF b) DE : EF c) EF : ED d) DF : EF

128. If the lengths of the corresponding sides BC and QR of two similar triangles ABC and PQR are respectively 6 cm and 10 cm, then the ratio of the areas of Δ ABC and Δ PQR is

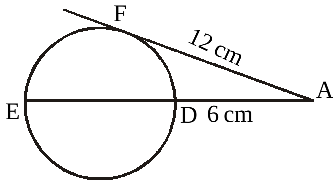
a) 3 : 5 b) 9 : 25 c) 25 : 9 d) 5 : 3

129. The corresponding sides of two similar triangles are in the ratio $a : b$. The ratio of their areas is

a) $a : b$ b) $2a : 2b$ c) $a^2 : b^2$ d) $\frac{1}{a} : \frac{1}{b}$

130. In the diagram, if AD = 6 cm, AF = 12 cm, then the length of DE =

a) 12 cm b) 24 cm
c) 18 cm d) 144 cm



131. Two chords AB and CD cut internally at E.
If AE = 6 cm, EB = 8 cm and EC = 4 cm, then ED is equal to

a) 14 cm b) 12 cm c) 10 cm d) 32 cm

132. Two circles of radii 8.2 cm and 3.6 cm touch each other externally, the distance between their centres is

a) 4.6 cm b) 11.8 cm c) 4.1 cm d) 1.8 cm

133. The distance between the centres of two circles is 13 cm and the radii are 8cm and 3 cm respectively. The length of their direct common tangent is

a) 8 cm b) 5 cm c) 13 cm d) 12 cm

134. The distance between the centres of two circles is 10 cm and the radii are 4cm and 2 cm respectively. The length of their transverse common tangent is

a) 6 cm b) 8 cm c) 12 cm d) 10 cm

135. In a Δ ABC, $\angle B = 90^o$ BD $\perp$ AC, AB = 3 cm, AC = 5cm, AD =

a) 9 cm b) 3.2 cm c) 1.8 cm d) 8.2 cm

8. CO-ORDINATE GEOMETRY

136. The area of a quadrilateral formed by the points (–1, 1), (1, 1), (1, –1) and (–1,–1) is

a) 0 sq.units b) 4 sq.units c) 25 sq.units d) –1sq.units

137. The area of a triangle formed by the points (0, 4), (4, 0) and origin is

a) 8 sq.units b) 16 sq.units c) 2 sq.units d) 4 sq.units

138. A man walks near a wall, such that the distance between him and the wall is 5 units. Considering the wall to be the *X*-axis, what is the equation of the path travelled by the man?

a) $x = 5$ b) y = 5 c) $x = 0$ d) y = 0

139. If $x - y = 3$ and $x + 2y = 6$ are the diameters of the circle, then the centre of the circle is

a) (0, 0) b) (2, 2) c) (1, –1) d) (4,1)

140. The straight line given by the equation $y = 5$ is

a) parallel to *x*-axis b) parallel to *y*-axis

c) passing through the origin d) perpendicular to *x*-axis

141. Find the angle between the lines x = y and $\sqrt{3}\,x - y = 0$

a) 15^0 b) 30^0 c) 60^0 d) 90^0

142. A straight road AB (A is in IV quadrant) is such that it bends at B (1, 0) by an angle of 30^0 towards the right. Considering the line perpendicular to AB through B to be X- axis, equations of the two parts of the road are

a) $x = 1$, $\sqrt{3}\,x - y - \sqrt{3} = 0$ b) y = 1, $x - \sqrt{3}\,y + 1 = 0$

c) $x = 0$, y = $\sqrt{3}$ c) y = 1, $x = 1$

143. If (5, 7), (3, *a*), (6, 6) are collinear, then the value of *a* is

a) 3 b) 6 c) 9 d) 12

144. The point of intersection of $3x - y = 2$ and $x + y = 6$ is

a) (4, 4) b) (4, 10) c) (10, 4) d) (2, 4)

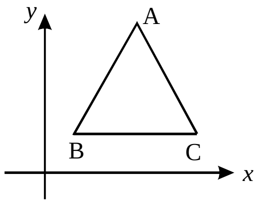
145. If the slope of the line joining (–6, 13) and (3, *a*) is $\frac{-1}{3}$, then the value of *a* is

a) 5 b) –10 c) –5 d) 10

146. The slope of the line which is perpendicular to the line joining the points (0, 0) and (–1, 1) is

a) 1 b) –1 c) $\frac{1}{2}$ d) –2

147. In the figure, the side BC of an equilateral triangle ABC is parallel to the *X*-axis. The slope of AB is



a) $\sqrt{3}$ b) $-\sqrt{3}$ c) $\frac{-1}{\sqrt{3}}$ d) not defined

148. If slope of the line AB is $\frac{1}{5}$, then the slope of perpendicular bisector of AB is

a) 5 b) $-\frac{1}{5}$ c) –5 d) 0

149. Slope of the line $2x + 3y + 6 = 0$ is

a) $\frac{2}{3}$ b) $-\frac{2}{3}$ c) –2 d) $\frac{1}{2}$

150. If A is a point on the Y axis whose ordinate is 4 and B is a point on the X axis whose abscissa is 3, then the equation of the line AB is

a) $3x + 4y = 12$ b) $4x + 3y = 12$ c) $3x - 4y = 0$ d) $3x - y + 12 = 0$

151. The value of p, given that the line $\frac{y}{2} = x - p$ passes through the point (–4, 4) is

a) 4 b) – 6 c) –2 d) 3

152. If lines $ax - 5y = 5$ and $2x + y = 1$ are perpendicular then the value of a is

a) 2 b) $\frac{5}{2}$ c) $\frac{2}{5}$ d) $\frac{1}{2}$

153. The X intercept of the line $4x - 7y + 28 = 0$ is

a) 7 b) –7 c) $\frac{1}{7}$ d) $-\frac{1}{7}$

154. (2, 1) is the point of intersection of the two lines

a) $x - y = 3$, $3x - y = 7$ b) $x + y = 3$, $3x + y = 7$
c) $3x + y = 3$, $x + y = 7$ d) $x + 3y = 3$, $x - y = 7$

155. The equation of the line passing through the origin and parallel to the line $3x + 2y - 5 = 0$ is

a) $3x - 2y + 5 = 0$ b) $2x + 3y = 0$ c) $3x + 2y = 0$ d) $2x - 3y = 0$

156. The line $2x - 5y - 10 = 0$ meets the Y-axis at

a) (0, 2) b) (0, –2) c) (2, 0) d) (–2, 0)

157. The equation of a straight line which has the Y intercept –5 and slope 2 is

a) $2x + y + 5 = 0$ b) $2x - y + 5 = 0$ c) $2x - y - 5 = 0$ d) $2x + y - 5 = 0$

158. The lines $y = -3$ and $x = 8$ meet at the point

a) (– 8, –3) b) (3, 8) c) (–3, 8) d) (8, –3)

9. TRIGONOMETRY

159. $\dfrac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}=$

a) 1 b) $1-\sin\theta\cos\theta$ c) $\sin\theta+\cos\theta$ d) $\tan\theta$

160. $\sin^2 1^o+\sin^2 2^o + \ldots + \sin^2 90^o =$

a) 90 b) 45 c) 46 d) 45.5

161. A circle is divided into *n* equal sectors. The tangent of each angle at the centre is

a) tan (*n*) b) $\tan\left(\dfrac{360^o}{n}\right)$ c) $\sqrt{3}$ d) $\dfrac{1}{\sqrt{3}}$

162. $\dfrac{\sqrt{1-\sin^2\theta}}{\sin\theta}=$

a) $\cot\theta$ b) $\dfrac{\sin\theta}{2}$ c) $\tan\theta$ d) $\dfrac{1+\sin\theta}{\sin^2\theta}$

163. $\sec\theta\sqrt{1-\sin^2\theta}=$

a) $\cos\theta$ b) 1 c) 0 d) $\sec\theta$

164. A trekker, before climbing a mountain finds the height of the mountain from a point 20 km from it. He finds the angle of elevation to be 30º. The height of the mountain is

a) $\dfrac{20\sqrt{3}}{3}$ km b) $20\sqrt{3}$ km c) 20 km d) 30 km

165. The values of $\tan\theta$ from the equation $3(\sec^2\theta-1)+16\tan\theta+5=0$ are

a) $\left\{\dfrac{1}{3},5\right\}$ b) $\left\{\dfrac{1}{5},3\right\}$ c) $\left\{-\dfrac{1}{3},-\dfrac{1}{5}\right\}$ d) $\left\{-\dfrac{1}{3},-5\right\}$

166. The simplified value of $1-\dfrac{\sin^2\theta}{1+\cos\theta}$ is

a) $\cos\theta$ b) $\sin\theta$ c) $\tan\theta$ d) $\operatorname{cosec}\theta$

167. If $(1-\cos^2\theta)=\dfrac{3}{4}$, then $\sin\theta=$

a) $\dfrac{\sqrt{3}}{2}$ b) $\dfrac{1}{2}$ c) 1 d) 0

168. If $\tan 40^o = x$, then the value of $1 + \text{cosec}^2 50^o$ is

a) $1+x^2$ b) $2+x^2$ c) x d) x^2

169. If $\tan 2\theta = \cot(\theta + 6^o)$ where 2θ and θ are actue angles, then the value of θ is

a) 36^o b) 26^o c) 28^o d) 6^o

170. $\dfrac{\sin 60^o}{\sin 30^o}(1+\cos 60^o) =$

a) $\sqrt{3}+\dfrac{\sqrt{3}}{2}$ b) $\dfrac{1}{\sqrt{3}}+\dfrac{1}{2}$ c) $\dfrac{5}{\sqrt{2}}+\dfrac{1}{2}$ d) $1+\dfrac{1}{\sqrt{2}}$

171. $2(\sin^2 60^o + \cos^2 30^o) - (\sin^2 45^o + \cos^2 45^o)$ is

a) 3 b) 2 c) 1 d) 0

172. $\dfrac{\sqrt{\text{cosec}^2 15^o - 1}}{\text{cosec } 15^o} =$

a) $\sin 15^o$ b) $\cos 15^o$ c) $2+\sqrt{3}$ d) $\dfrac{-1-\sqrt{2}}{2}$

173. $(\tan^2\theta - \sin^2\theta)(\cot^2\theta - \cos^2\theta) =$

a) 1 b) $\sin^2\theta \cos^2\theta$ c) 0 d) $\tan^2\theta - \cot^2\theta$

174. The value of $\tan^2\theta - \dfrac{1}{\cos^2\theta} =$

a) 1 b) -1 c) 2 d) -2

175. The value of $\sin^2 18^o + \sin^2 72^o$ is

a) -1 b) 18 c) 72 d) 1

176. $\sin^2 A \cos^2 B - \cos^2 A \sin^2 B$ is

a) $\sin^2 A$ b) $\sin^2 B$ c) $\sin^2 A - \sin^2 B$ d) $\sin^2 A - \cos^2 B$

177. If $\cos A = \dfrac{1}{2}$, then the value of $\sin^2 A + \cos^2 A$ is

a) 2 b) 1 c) -1 d) $\dfrac{1}{2}$

178. If $\tan\theta + \cot\theta = 2$ then the value of $\tan^2\theta + \cot^2\theta$ is

a) 0 b) 1 c) 2 d) 4

179. When the angle of elevation of the sun is 45^o, the length of the shadow of a tower of height 10m is

a) 10 m b) $10\sqrt{3}$ c) $\dfrac{10}{\sqrt{3}}$ d) $\dfrac{1}{\sqrt{3}}$

180. The value of $\sec^2 30^\circ + \operatorname{cosec}^2 30^\circ - \cos^2 45^\circ + 3\cot^2 60^\circ$ is

a) $\frac{5}{6}$ b) $5\frac{5}{6}$ c) $\frac{30}{6}$ d) $\frac{42}{9}$

181. $(\tan 7^\circ)(\tan 23^\circ)(\tan 60^\circ)(\tan 67^\circ)(\tan 83^\circ)$ is

a) 0 b) 7 c) 1 d) $\sqrt{3}$

182. $35^\circ - 30^\circ 17' 20''$

a) $65^\circ 17' 20''$ b) $4^\circ 42' 40''$ c) $5^\circ 43' 40''$ d) $6^\circ 42' 40''$

11. STATISTICS

183. The range of the first 5 prime natural numbers in order is

a) 6 b) 9 c) 5 d) 82

184. The range of the following set of values is

x	45	55	65	75	85
f	4	2	5	2	1

a) 3 b) 4 c) 40 d) 65

185. The variance of the first 7 natural numbers is

a) 5 b) 4 c) 16 d) 8

186. The range of the first 20 odd natural numbers is

a) 38 b) 40 c) 19 d) 39

187. If the standard deviation for a set of data is 0.3, then the variance is

a) 0.9 b) 0.09 c) 9 d) 0.0009

188. The standard deviation is the ………… of the variance.

a) cube b) square c) square root d) cube root

189. The standard deviation of 5 values is $5\sqrt{2}$. If each value is increased by 4, then the new standard deviation is

a) $20\sqrt{4}$ b) $10\sqrt{2}$ c) $5\sqrt{2}$ d) $\frac{5}{2}\sqrt{2}$

190. The variance of 5 values is 16. If each value is doubled, then the standard deviation of new value is

a) 4 b) 8 c) 32 d) 16

191. The probability of getting a sum of 13 in the rolling a die twice is

a) 1 b) $\frac{1}{6}$ c) $\frac{1}{36}$ d) 0

192. The probability of getting neither an ace nor a king from a pack of 52 cards is

a) $\frac{2}{13}$ b) $\frac{11}{13}$ c) $\frac{4}{13}$ d) $\frac{8}{13}$

193. If $P(A) = 0.37$, $P(B) = 0.42$ and $P(A \cap B) = 0.09$, then $P(A \cup B)$

a) 0.6 b) 0.7 c) 0.8 d) 0.9

194. The probability of selecting a queen of hearts when a card is drawn from a well shuffled pack of 52 cards is

a) $\frac{1}{52}$ b) $\frac{16}{52}$ c) $\frac{1}{13}$ d) $\frac{2}{52}$

195. A fair die is thrown once. The probability of getting a prime or composite number is

a) 1 b) 0 c) $\frac{5}{6}$ d) $\frac{1}{6}$

196. The probability of an impossible event is

a) 0 b) 1 c) 5 d) 2

197. The probability of getting 3 heads or 3 tails in tossing 3 coins is

a) $\frac{1}{8}$ b) $\frac{1}{4}$ c) $\frac{3}{8}$ d) $\frac{1}{2}$

198. A scientist experimented with pea plants. After crossing two plants that are tall and dwarf, the ratio of number of plants obtained in the first generation is 3 : 1 (tall : dwarf). If the scientist experimented and got 576 plants in first generation, what is the probability that the plant he chooses for the next generation is tall?

a) $\frac{3}{4}$ b) $\frac{1}{4}$ c) $\frac{1}{3}$ d) $\frac{144}{576}$

199. The probability of a sure event is

a) 1 b) 100 c) 0 d) 0.1

200. What is the probability of drawing the number '6' from a well shuffled pack of 52 cards?

a) $\frac{1}{13}$ b) $\frac{1}{2}$ c) $\frac{1}{26}$ d) $\frac{1}{4}$

ANSWERS TO OBJECTIVE TYPE QUESTIONS

1.	(d)	2.	(d)	3.	(c)	4.	(c)	5.	(d)	6.	(a)
7.	(b)	8.	(b)	9.	(b)	10.	(c)	11.	(a)	12.	(a)
13.	(b)	14.	(a)	15.	(a)	16.	(a)	17.	(d)	18.	(b)
19.	(b)	20.	(b)	21.	(c)	22.	(a)	23.	(b)	24.	(c)
25.	(d)	26.	(c)	27.	(a)	28.	(b)	29.	(d)	30.	(d)
31.	(b)	32.	(a)	33.	(b)	34.	(b)	35.	(b)	36.	(a)
37.	(b)	38.	(b)	39.	(c)	40.	(b)	41.	(d)	42.	(a)
43.	(b)	44.	(b)	45.	(a)	46.	(c)	47.	(c)	48.	(c)
49.	(d)	50.	(c)	51.	(c)	52.	(c)	53.	(a)	54.	(a)
55.	(b)	56.	(b)	57.	(b)	58.	(b)	59.	(d)	60.	(b)
61.	(c)	62.	(b)	63.	(c)	64.	(b)	65.	(b)	66.	(a)
67.	(c)	68.	(a)	69.	(c)	70.	(a)	71.	(b)	72.	(a)
73.	(d)	74.	(b)	75.	(c)	76.	(b)	77.	(c)	78.	(d)
79.	(d)	80.	(b)	81.	(c)	82.	(d)	83.	(b)	84.	(d)
85.	(c)	86.	(d)	87.	(d)	88.	(c)	89.	(b)	90.	(c)
91.	(c)	92.	(a)	93.	(a)	94.	(a)	95.	(d)	96.	(c)
97.	(d)	98.	(b)	99.	(a)	100.	(c)	101.	(d)	102.	(b)
103.	(c)	104.	(a)	105.	(c)	106.	(b)	107.	(b)	108.	(b)
109.	(d)	110.	(c)	111.	(d)	112.	(a)	113.	(d)	114.	(a)
115.	(b)	116.	(b)	117.	(b)	118.	(c)	119.	(d)	120.	(d)
121.	(a)	122.	(a)	123.	(c)	124.	(a)	125.	(b)	126.	(b)
127.	(c)	128.	(b)	129.	(c)	130.	(c)	131.	(b)	132.	(b)
133.	(d)	134.	(b)	135.	(c)	136.	(b)	137.	(a)	138.	(b)
139.	(d)	140.	(a)	141.	(a)	142.	(a)	143.	(c)	144.	(d)
145.	(d)	146.	(a)	147.	(a)	148.	(c)	149.	(b)	150.	(b)
151.	(b)	152.	(b)	153.	(b)	154.	(b)	155.	(c)	156.	(b)
157.	(c)	158.	(d)	159.	(b)	160.	(d)	161.	(b)	162.	(a)
163.	(b)	164.	(a)	165.	(d)	166.	(a)	167.	(a)	168.	(b)
169.	(c)	170.	(a)	171.	(b)	172.	(b)	173.	(b)	174.	(b)
175.	(d)	176.	(c)	177.	(b)	178.	(c)	179.	(a)	180.	(b)
181.	(d)	182.	(b)	183.	(b)	184.	(c)	185.	(b)	186.	(a)
187.	(b)	188.	(c)	189.	(c)	190.	(b)	191.	(d)	192.	(b)
193.	(b)	194.	(a)	195.	(c)	196.	(a)	197.	(b)	198.	(a)
199.	(a)	200.	(a)								